ĐỀ THI VÀ ĐÁP ÁN CHI TIẾT
CỦA THI VÔ ĐỊCH TOÁN C P TRUNG H CỨC M R NG NĂM 2015
1. A number written in base $a$ is $123_a$. The same number written in base $b$ is $146_b$. What is the minimum value of $a + b$? [2 marks]

2. A circle is inscribed in a hexagon $ABCDEF$ so that each side of the hexagon is tangent to the circle. Find the perimeter of the hexagon if $AB = 6$, $CD = 7$, and $EF = 8$. [2 marks]

3. A selection of 3 whatsits, 7 doovers and 1 thingy cost a total of $329. A selection of 4 whatsits, 10 doovers and 1 thingy cost a total of $441. What is the total cost, in dollars, of 1 whatsit, 1 doover and 1 thingy? [3 marks]

4. A fraction, expressed in its lowest terms $\frac{a}{b}$, can also be written in the form $\frac{2}{n} + \frac{1}{n^2}$, where $n$ is a positive integer. If $a + b = 1024$, what is the value of $a$? [3 marks]

5. Determine the smallest positive integer $y$ for which there is a positive integer $x$ satisfying the equation $2^{13} + 2^{10} + 2^x = y^2$. [3 marks]

6. The large circle has radius $\frac{30}{\sqrt{\pi}}$. Two circles with diameter $\frac{30}{\sqrt{\pi}}$ lie inside the large circle. Two more circles lie inside the large circle so that the five circles touch each other as shown. Find the shaded area. [4 marks]

7. Consider a shortest path along the edges of a $7 \times 7$ square grid from its bottom-left vertex to its top-right vertex. How many such paths have no edge above the grid diagonal that joins these vertices? [4 marks]

8. Determine the number of non-negative integers $x$ that satisfy the equation

$$\left\lfloor \frac{x}{44} \right\rfloor = \left\lfloor \frac{x}{45} \right\rfloor.$$ 

(Note: if $r$ is any real number, then $\lfloor r \rfloor$ denotes the largest integer less than or equal to $r$.) [4 marks]
9. A sequence is formed by the following rules: \( s_1 = a, s_2 = b \) and \( s_{n+2} = s_{n+1} + (-1)^n s_n \) for all \( n \geq 1 \).

If \( a = 3 \) and \( b \) is an integer less than 1000, what is the largest value of \( b \) for which 2015 is a member of the sequence? Justify your answer. [5 marks]

10. \( X \) is a point inside an equilateral triangle \( ABC \). \( Y \) is the foot of the perpendicular from \( X \) to \( AC \), \( Z \) is the foot of the perpendicular from \( X \) to \( AB \), and \( W \) is the foot of the perpendicular from \( X \) to \( BC \).

The ratio of the distances of \( X \) from the three sides of the triangle is 1 : 2 : 4 as shown in the diagram.

If the area of \( AZXY \) is 13 cm\(^2\), find the area of \( ABC \). Justify your answer. [5 marks]

**Investigation**

If \( XY : XZ : XW = a : b : c \), find the ratio of the areas of \( AZXY \) and \( ABC \). [2 bonus marks]
1. **Method 1**

\[ 123_a = 146_b \iff a^2 + 2a + 3 = b^2 + 4b + 6 \]
\[ \iff (a + 1)^2 + 2 = (b + 2)^2 \]
\[ \iff a + 1 = b + 2 \ (a \text{ and } b \text{ are positive}) \]
\[ \iff a = b + 1 \]

Since the minimum value for \( b \) is 7, the minimum value for \( a + b \) is \( 8 + 7 = 15 \).

**Method 2**

Since the digits in any number are less than the base, \( b \geq 7 \).

We also have \( a > b \), otherwise \( a^2 + 2a + 3 < b^2 + 4b + 6 \).

If \( b = 7 \) and \( a = 8 \), then \( a^2 + 2a + 3 = 83 = b^2 + 4b + 6 \).

So the minimum value for \( a + b \) is \( 8 + 7 = 15 \).

2. Let \( AB, BC, CD, DE, EF, FA \) touch the circle at \( U, V, W, X, Y, Z \) respectively.

Since the two tangents from a point to a circle have equal length, \( UB = BV, VC = CW, WD = DX, XE = EY, YF = FZ, ZA = AU \).

The perimeter of hexagon \( ABCDEF \) is

\[ AU + UB + BV + VC + CW + WD + DX + XE + EY + YF + FZ + ZA \]
\[ = AU + UB + UB + CW + CW + WD + WD + EY + EY + YF + YF + AU \]
\[ = 2(UA + UB + CW + WD + EY + YF) \]
\[ = 2(AB + CD + EF) = 2(6 + 7 + 8) = 2(21) = 42. \]

3. **Preamble**

Let the required cost be \( x \). Then, with obvious notation, we have:

\[ 3w + 7d + t = 329 \quad (1) \]
\[ 4w + 10d + t = 441 \quad (2) \]
\[ w + d + t = x \quad \text{(3)} \]

**Method 1**

\[ 3 \times (1) - 2 \times (2): w + d + t = 3 \times 329 - 2 \times 441 = 987 - 882 = 105. \]
Method 2

(2) − (1): \( w + 3d = 112. \)
(1) − (3): \( 2w + 6d = 329 - x = 2 \times 112 = 224. \)

Then \( x = 329 - 224 = 105. \)

Method 3

\[ 10 \times (1) - 7 \times (2): w = \frac{203 - 3t}{2} \]
\[ 3 \times (2) - 4 \times (1): d = \frac{7 + t}{2} \]

Then \( w + d + t = \frac{210}{2} - 2t/2 + t = 105. \)

4. We have \( \frac{2}{n} + \frac{1}{n^2} = \frac{2n + 1}{n^2}. \)

Since \( 2n + 1 \) and \( n^2 \) are coprime, \( a = 2n + 1 \) and \( b = n^2. \)

So 1024 = \( a + b = n^2 + 2n + 1 = (n + 1)^2 \), hence \( n + 1 = 32. \)

This gives \( a = 2n + 1 = 2 \times 31 + 1 = 63. \)

5. Method 1

\[ 2^{13} + 2^{10} + 2^x = y^2 \]
\[ \iff 2^{10}(2^3 + 1) + 2^x = y^2 \]
\[ \iff (2^5 \times 3)^2 + 2^x = y^2 \]
\[ \iff 2^x = y^2 - 96^2 \]
\[ \iff 2^x = (y + 96)(y - 96). \]

Since \( y \) is an integer, both \( y + 96 \) and \( y - 96 \) must be powers of 2.

Let \( y + 96 = 2^m \) and \( y - 96 = 2^n. \) Then \( 2^m - 2^n = 192 = 2^6 \times 3. \)

Hence \( 2^{m-6} - 2^{n-6} = 3. \) So \( 2^{m-6} = 4 \) and \( 2^{n-6} = 1. \)

In particular, \( m = 8. \) Hence \( y = 2^8 - 96 = 256 - 96 = 160. \)

Method 2

We have \( y^2 = 2^{13} + 2^{10} + 2^x = 2^{10}(2^3 + 1 + 2^{x-10}) = 2^{10}(9 + 2^{x-10}). \)

So we want the smallest value of \( 9 + 2^{x-10} \) that is a perfect square.

Since \( 9 + 2^{x-10} \) is odd and greater than 9, \( 9 + 2^{x-10} \geq 25. \)

Since \( 9 + 2^{14-10} = 25, \) \( y = 2^5 \times 5 = 32 \times 5 = 160. \)

Comment

Method 1 shows that \( 2^{13} + 2^{10} + 2^x = y^2 \) has only one solution.

6. The centres \( Y \) and \( Y' \) of the two medium circles lie on a diameter of the large circle. By symmetry about this diameter, the two smaller circles are congruent. Let \( X \) be the centre of the large circle and \( Z \) the centre of a small circle.

Let \( R \) and \( r \) be the radii of a medium and small circle respectively. Then \( ZY = R + r = ZY'. \) Since \( XY = XY', \) triangles \( XYZ \) and \( XY'Z \) are congruent. Hence \( XZ \perp XY. \)
By Pythagoras, $YZ^2 = YX^2 + XZ^2$. So $(R + r)^2 = R^2 + (2R - r)^2$.
Then $R^2 + 2Rr + r^2 = 5R^2 - 4Rr + r^2$, which simplifies to $3r = 2R$.
So the large circle has area $\pi (30/\sqrt{\pi})^2 = 900$, each medium circle has area $\pi (15/\sqrt{\pi})^2 = 225$, and each small circle has area $\pi (10/\sqrt{\pi})^2 = 100$.
Thus the shaded area is $900 - 2 \times 225 - 2 \times 100 = 250$.

7. **Method 1**

Any path from the start vertex $O$ to a vertex $A$ must pass through either the vertex $L$ left of $A$ or the vertex $U$ underneath $A$. So the number of paths from $O$ to $A$ is the sum of the number of paths from $O$ to $L$ and the paths from $O$ to $U$.

There is only one path from $O$ to any vertex on the bottom line of the grid. So the number of paths from $O$ to all other vertices can be progressively calculated from the second bottom row upwards as indicated.

Thus the number of required paths is **429**.

**Method 2**

To help understand the problem, consider some smaller grids.

Let $p(n)$ equal the number of required paths on an $n \times n$ grid and let $p(0) = 1$. 
Starting with the bottom-left vertex, label the vertices of the diagonal 0, 1, 2, \ldots, \(n\).

Consider all the paths that touch the diagonal at vertex \(i\) but not at any of the vertices between vertex 0 and vertex \(i\). Each such path divides into two subpaths.

One subpath is from vertex 0 to vertex \(i\) and, except for the first and last edge, lies in the lower triangle of the diagram above. Thus there are \(p(i-1)\) of these subpaths.

The other subpath is from vertex \(i\) to vertex \(n\) and lies in the upper triangle in the diagram above. Thus there are \(p(n-i)\) of these subpaths.

So the number of such paths is \(p(i-1) \times p(n-i)\).

Summing these products from \(i = 1\) to \(i = n\) gives all required paths. Thus

\[
p(n) = p(n-1) + p(1)p(n-2) + p(2)p(n-3) + \cdots + p(n-2)p(1) + p(n-1)
\]

We have \(p(1) = 1\), \(p(2) = 2\), \(p(3) = 5\). So

\[
p(4) = p(3) + p(1)p(2) + p(3)p(1) + p(3) = 14,
p(5) = p(4) + p(1)p(3) + p(2)p(2) + p(3)p(1) + p(4) = 42,
p(6) = p(5) + p(1)p(4) + p(2)p(3) + p(3)p(2) + p(1)p(4) + p(5) = 132, \text{ and}
p(7) = p(6) + p(1)p(5) + p(2)p(4) + p(3)p(3) + p(4)p(2) + p(5)p(1) + p(6) = 429.
\]

8. **Method 1**

Let \( \left\lfloor \frac{x}{44} \right\rfloor = \left\lfloor \frac{x}{45} \right\rfloor = n \).

Since \(x\) is non-negative, \(n\) is also non-negative.

If \(n = 0\), then \(x\) is any integer from 0 to 44 - 1 = 43: a total of 44 values.

If \(n = 1\), then \(x\) is any integer from 45 to 2 \times 44 - 1 = 87: a total of 43 values.

If \(n = 2\), then \(x\) is any integer from 2 \times 45 = 90 to 3 \times 44 - 1 = 131: a total of 42 values.

If \(n = k\), then \(x\) is any integer from 45k to 44(k + 1) - 1 = 44k + 43: a total of (44k + 43) - (45k - 1) = 44 - k values.

Thus, increasing \(n\) by 1 decreases the number of values of \(x\) by 1. Also the largest value of \(n\) is 43, in which case \(x\) has only 1 value.

Therefore the number of non-negative integer values of \(x\) is 44 + 43 + \cdots + 1 = \frac{1}{2}(44 \times 45) = 990.

**Method 2**

Let \(n\) be a non-negative integer such that \( \left\lfloor \frac{x}{44} \right\rfloor = \left\lfloor \frac{x}{45} \right\rfloor = n \).

Then \( \left\lfloor \frac{x}{44} \right\rfloor = n \iff 44n \leq x < 44(n+1) \) and \( \left\lfloor \frac{x}{45} \right\rfloor = n \iff 45n \leq x < 45(n+1) \).

So \( \left\lfloor \frac{x}{44} \right\rfloor = \left\lfloor \frac{x}{45} \right\rfloor = n \iff 45n \leq x < 44(n+1) \iff 44n + n \leq x < 44n + 44 \).

This is the case if and only if \(n < 44\), and then \(x\) can assume exactly 44 - \(n\) different values.

Therefore the number of non-negative integer values of \(x\) is

\[
(44 - 0) + (44 - 1) + \cdots + (44 - 43) = 44 + 43 + \cdots + 1 = \frac{1}{2}(44 \times 45) = 990.
\]
Method 3

Let $n$ be a non-negative integer such that $\left\lfloor \frac{x}{44} \right\rfloor = \left\lfloor \frac{x}{45} \right\rfloor = n$.

Then $x = 44n + r$ where $0 \leq r \leq 43$ and $x = 45n + s$ where $0 \leq s \leq 44$.

So $n = r - s$. Therefore $0 \leq n \leq 43$. Also $r = n + s$. Therefore $n \leq r \leq 43$.

Therefore the number of non-negative integer values of $x$ is $44 + 43 + \cdots + 1 = \frac{1}{2}(44 \times 45) = 990$.

9. Working out the first few terms gives us an idea of how the given sequence develops:

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<th>$s_{2n-1}$</th>
<th>$s_{2n}$</th>
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<td>$b$</td>
</tr>
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<td>$b - a$</td>
<td>$2b - a$</td>
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<td>$b$</td>
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<td>4</td>
<td>$2b - a$</td>
<td>$5b - 2a$</td>
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<td>5</td>
<td>$3b - a$</td>
<td>$8b - 3a$</td>
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<td>6</td>
<td>$5b - 2a$</td>
<td>$13b - 5a$</td>
</tr>
<tr>
<td>7</td>
<td>$8b - 3a$</td>
<td>$21b - 8a$</td>
</tr>
</tbody>
</table>

It appears that the coefficients in the even terms form a Fibonacci sequence and, from the 5th term, every odd term is a repeat of the third term before it.

These observations are true for the entire sequence since, for $m \geq 1$, we have:

\[
\begin{align*}
    s_{2m+2} &= s_{2m+1} + s_{2m} \\
    s_{2m+3} &= s_{2m+2} - s_{2m+1} = s_{2m} \\
    s_{2m+4} &= s_{2m+3} + s_{2m+2} = s_{2m+2} + s_{2m}
\end{align*}
\]

So, defining $F_1 = 1$, $F_2 = 2$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$, we have $s_{2n} = bF_n - aF_{n-2}$ for $n \geq 3$. Since $a = 3$ and $b < 1000$, none of the first five terms of the given sequence equal 2015. So we are looking for integer solutions of $bF_n - 3F_{n-2} = 2015$ for $n \geq 3$.

$s_6 = 3b - 3 = 2015$, has no solution.

$s_8 = 5b - 6 = 2015$, has no solution.

$s_{10} = 8b - 9 = 2015$ implies $b = 253$.

For $n \geq 6$ we have $b = \frac{2015/F_n + 3F_{n-2}/F_n}{F_n}$ since $F_n$ increases, we have $F_n \geq 13$ and $F_{n-2}/F_n < 1$ for $n \geq 6$.

Hence $b < 2015/13 + 3 = 158$. So the largest value of $b$ is 253.

10. Method 1

We first show that $X$ is uniquely defined for any given equilateral triangle $ABC$.

Let $P$ be a point outside $\triangle ABC$ such that its distances from $AC$ and $AB$ are in the ratio $1:2$. By similar triangles, any point on the line $AP$ has the same property. Also any point between $AP$ and $AC$ has the distance ratio less than $1:2$ and any point between $AP$ and $AB$ has the distance ratio greater than $1:2$. 

![Diagram of equilateral triangle ABC with point P and distances Q1, 1, A4, 2, B2, 4, C1, 1]
Let \( Q \) be a point outside \( \triangle ABC \) such that its distances from \( AC \) and \( BC \) are in the ratio 1:4. By an argument similar to that in the previous paragraph, only the points on \( CQ \) have the distance ratio equal to 1:4. Thus the only point whose distances to \( AC, AB, \) and \( BC \) are in the ratio 1:2:4 is the point \( X \) at which \( AP \) and \( CQ \) intersect.

Scaling if necessary, we may assume that the actual distance \( s \) of \( X \) to the sides of \( \triangle ABC \) are 1, 2, 4. Let \( h \) be the height of \( \triangle ABC \). Letting \( \text{area} \) denote area, we have

\[
|\triangle ABC| = \frac{1}{2} h \times AB \quad \text{and} \quad |\triangle ABC| = |\triangle AXB| + |\triangle BXC| + |\triangle CXA| = \frac{1}{2}(2AB + 4BC + AC) = \frac{1}{2}AB \times 7.
\]

So \( h = 7 \).

Draw a 7-layer grid of equilateral triangles each of height 1, starting with a single triangle in the top layer, then a trapezium of 3 triangles in the next layer, a trapezium of 5 triangles in the next layer, and so on. The boundary of the combined figure is \( \triangle ABC \) and \( X \) is one of the grid vertices as shown.

There are 49 small triangles in \( \triangle ABC \) and 6.5 small triangles in \( AZXY \). Hence, after rescaling so that the area of \( AZXY \) is 13 cm\(^2\), the area of \( \triangle ABC \) is 13 \times 49/6.5 = 98 cm\(^2\).

**Method 2**

Join \( AX, BX, CX \). Since \( \angle YAX = \angle ZBW = 60^\circ \), the quadrilaterals \( AZXY \) and \( BWXZ \) are similar. Let \( XY \) be 1 unit and \( AY \) be \( x \). Then \( BZ = 2x \).

By Pythagoras: in \( \triangle AXY \), \( AX = \sqrt{1 + x^2} \) and in \( \triangle AXZ \), \( AZ = \sqrt{x^2 - 3} \). Hence \( BW = 2\sqrt{x^2 - 3} \).

Since \( AB = AC, YC = x + \sqrt{x^2 - 3} \).

By Pythagoras: in \( \triangle XYZ \), \( XC^2 = 1 + (x + \sqrt{x^2 - 3})^2 = 2x^2 - 2 + 2x\sqrt{x^2 - 3} \)
and in \( \triangle XWC \), \( WC^2 = 2x^2 - 18 + 2x\sqrt{x^2 - 3} \).

Since \( BA = BC, 2x + \sqrt{x^2 - 3} = 2\sqrt{x^2 - 3} + \sqrt{2x^2 - 18 + 2x\sqrt{x^2 - 3}} \).

So \( 2x - \sqrt{x^2 - 3} = \sqrt{2x^2 - 18 + 2x\sqrt{x^2 - 3}} \).
Squaring gives $4x^2 + x^2 - 3 - 4x\sqrt{x^2 - 3} = 2x^2 - 18 + 2x\sqrt{x^2 - 3}$, which simplifies to $3x^2 + 15 = 6x\sqrt{x^2 - 3}$.

Squaring again gives $9x^4 + 90x^2 + 225 = 36x^4 - 108x^2$. So $0 = 3x^4 - 22x^2 - 25 = (3x^2 - 25)(x^2 + 1)$, giving $x = \frac{5}{\sqrt{3}}$.

Hence, area $AZXY = \frac{x}{2} + \sqrt{x^2 - 3} = \frac{5}{2\sqrt{3}} + \frac{4}{\sqrt{3}} = \frac{13}{2\sqrt{3}}$ and area $ABC = \frac{\sqrt{3}}{4}(2x + \sqrt{x^2 - 3})^2 = \frac{\sqrt{3}}{4}\left(\frac{10}{\sqrt{3}} + \frac{4}{\sqrt{3}}\right)^2 = \frac{49}{\sqrt{3}}$.

Since the area of $AZXY$ is $13\text{ cm}^2$, the area of $ABC$ is $2 \times 49 = 98\text{ cm}^2$.

Method 3

Let $DI$ be the line through $X$ parallel to $AC$ with $D$ on $AB$ and $I$ on $BC$.
Let $EG$ be the line through $X$ parallel to $BC$ with $E$ on $AB$ and $G$ on $AC$.
Let $FH$ be the line through $X$ parallel to $AB$ with $F$ on $AC$ and $H$ on $BC$.
Let $J$ be a point on $AB$ so that $HJ$ is parallel to $AC$.
Triangles $XDE$, $XFG$, $XHI$, $BHJ$ are equilateral, and triangles $XDE$ and $BHJ$ are congruent.

The areas of the various equilateral triangles are proportional to the square of their heights. Let the area of $\triangle F X G = 1$. Then, denoting area by $| |$, we have:

$|DEX| = 4$, $|XHI| = 16$, $|AEG| = 9$, $|DBI| = 36$, $|FHC| = 25$.
$|ABC| = |AEG| + |FHC| + |DBI| - |FXG| - |DEX| - |XHI| = 9 + 25 + 36 - 1 - 4 - 16 = 49$.
$|AZXY| = |AEG| - \frac{1}{2}(|FXG| + |DEX|) = 9 - \frac{1}{2}(1 + 4) = 6.5$.

Since the area of $AZXY$ is $13\text{ cm}^2$, the area of $ABC$ is $2 \times 49 = 98\text{ cm}^2$.

Method 4

Consider the general case where $XY = a$, $XZ = b$, and $XW = c$. 
Projecting $AY$ onto the line through $ZX$ gives $AY \sin 60^\circ - a \cos 60^\circ = b$. Hence $AY = (a + 2b)/\sqrt{3}$. Similarly, $AZ = (b + 2a)/\sqrt{3}$.

Letting $||$ denote area, we have

$$|AZXY| = |YAZ| + |YZX| = \frac{1}{2}(AY)(AZ) \sin 60^\circ + \frac{1}{2}ab \sin 120^\circ = \frac{\sqrt{3}}{2}((AY)(AZ) + ab) = \frac{\sqrt{3}}{2}((a + 2b)(b + 2a) + 3ab) = \frac{\sqrt{3}}{2}(2a^2 + 2b^2 + 8ab).

Similarly, $|CYXW| = \frac{\sqrt{3}}{6}(a^2 + b^2 + 4ab)$ and $|BWXZ| = \frac{\sqrt{3}}{6}(b^2 + c^2 + 4bc)$.

Hence $|ABC| = \frac{\sqrt{3}}{6}(2a^2 + 2b^2 + 2c^2 + 4ab + 4ac + 4bc) = \frac{\sqrt{3}}{6}(a + b + c)^2$.

So $|ABC|/|AZXY| = 2(a + b + c)^2/(a^2 + b^2 + 4ab)$.

Letting $a = k$, $b = 2k$, $c = 4k$, and $|AZXY| = 13 \text{ cm}^2$, we have $|ABC| = 26(49k^2)/(k^2 + 4k^2 + 8k^2) = 98 \text{ cm}^2$.

**Investigation**

Method 4 gives $|ABC|/|AZXY| = 2(a + b + c)^2/(a^2 + b^2 + 4ab)$.

Alternatively, as in Method 3,

$$|ABC| = |AEG| + |FHC| + |DBI| - |FXG| - |DEX| - |XHI| = (a + b)^2 + (a + c)^2 + (b + c)^2 - a^2 - b^2 - c^2 = (a + b + c)^2.$$

Also $|AZXY| = |AEG| - \frac{1}{4}(|FXG| + |DEX|) = (a + b)^2 - \frac{1}{4}(a^2 + b^2) = 2ab + \frac{1}{2}(a^2 + b^2)$.

So $|ABC|/|AZXY| = 2(a + b + c)^2/(a^2 + b^2 + 4ab)$. 
B NG PHÂN TÍCH K T QU CHI TI T VÀ DANH SÁCH H C SINH Đ T CH NG NH N "GI I TH NG" VÀ "XU T S C"
NĂM 2015
### Distribution of Awards/School Year

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<td>Shore School, NSW</td>
<td>10</td>
<td>26</td>
</tr>
<tr>
<td>Xinlu Xu</td>
<td>Presbyterian Ladies' College, Sydney, NSW</td>
<td>10</td>
<td>26</td>
</tr>
<tr>
<td>Jason (Yi) Yang</td>
<td>James Ruse Agricultural High School, NSW</td>
<td>8</td>
<td>26</td>
</tr>
<tr>
<td>Shukai Zhang</td>
<td>Hwa Chong Institution, Singapore</td>
<td>10</td>
<td>26</td>
</tr>
<tr>
<td>Yanjun Zhang</td>
<td>Hwa Chong Institution, Singapore</td>
<td>9</td>
<td>26</td>
</tr>
<tr>
<td>Kevin Zhu</td>
<td>James Ruse Agricultural High School, NSW</td>
<td>10</td>
<td>26</td>
</tr>
<tr>
<td>Jonathan Zuk</td>
<td>Elwood College, VIC</td>
<td>8</td>
<td>26</td>
</tr>
</tbody>
</table>
ĐỀ THI VÀ ĐÁP ÁN CHI TI T T

CU C THI VÔ Đ CH TOÁN C P TRUNG H C ÚC M R NG NĂM 2014
1. In base $b$, the square of $24_b$ is $521_b$. Find the value of $b$ in base 10.

   \[2 \text{ marks}\]

2. Triangles $ABC$ and $XYZ$ are congruent right-angled isosceles triangles. Squares $KLMB$ and $PQRS$ are as shown. If the area of $KLMB$ is 189, find the area of $PQRS$.

   \[2 \text{ marks}\]

3. Let $x$ and $y$ be positive integers that simultaneously satisfy the equations $xy = 2048$ and $\frac{x}{y} - \frac{y}{x} = 7.875$. Find $x$.

   \[3 \text{ marks}\]

4. Joel has a number of blocks, all with integer weight in kilograms. All the blocks of one colour have the same weight and blocks of a different colour have different weights.

   Joel finds that various collections of some of these blocks have the same total weight $w$ kg. These collections include:
   1. 5 red, 3 blue and 5 green;
   2. 4 red, 5 blue and 4 green;
   3. 7 red, 4 blue and some green.

   If $30 < w < 50$, what is the total weight in kilograms of 6 red, 7 blue and 3 green blocks?

   \[3 \text{ marks}\]

5. Let $\frac{1}{a} + \frac{1}{b} = \frac{1}{20}$, where $a$ and $b$ are positive integers. Find the largest value of $a + b$.

   \[4 \text{ marks}\]

PLEASE TURN OVER THE PAGE FOR QUESTIONS 6, 7, 8, 9 AND 10
6. Justin’s sock drawer contains only identical black socks and identical white socks, a total of less than 50 socks altogether.

If he withdraws two socks at random, the probability that he gets a pair of the same colour is 0.5. What is the largest number of black socks he can have in his drawer?

[4 marks]

7. A code is a sequence of 0s and 1s that does not have three consecutive 0s. Determine the number of codes that have exactly 11 digits.

[4 marks]

8. Determine the largest integer $n$ which has at most three digits and equals the remainder when $n^2$ is divided by 1000.

[4 marks]

9. Let $ABCD$ be a trapezium with $AB \parallel CD$ such that

(i) its vertices $A$, $B$, $C$, $D$, lie on a circle with centre $O$,
(ii) its diagonals $AC$ and $BD$ intersect at point $M$ and $\angle AMD = 60^\circ$,
(iii) $MO = 10$.

Find the difference between the lengths of $AB$ and $CD$.

[5 marks]

10. An $n \times n$ grid with $n > 1$ is covered by several copies of a $2 \times 2$ square tile as in the figure below. Each tile covers precisely four cells of the grid and each cell of the grid is covered by at least one cell of one tile. The tiles may be rotated 90 degrees.

(a) Show there exists a covering of the grid such that there are exactly $n$ black cells visible.
(b) Prove there is no covering where there are less than $n$ black cells visible.
(c) Determine the maximum number of visible black cells.

[4 marks]

Investigation

(i) Show that, for each possible pattern of 3 black cells and 6 white cells on a $3 \times 3$ grid, there is a covering whose visible cells have that pattern. [1 bonus mark]
(ii) Explain why not all patterns of 4 black cells and 12 white cells on a $4 \times 4$ grid can be achieved with a covering in which each new tile must be placed on top of all previous tiles that it overlaps. [1 bonus mark]
(iii) Determine the maximum number of visible black cells for a covering of an $n \times m$ grid where $1 < n < m$. [2 bonus marks]
Australian Intermediate Mathematics Olympiad 2014

Solutions

1. We have $24_b = 2b + 4$, $521_b = 5b^2 + 2b + 1$ and $521_b = (2b + 4)^2 = 4b^2 + 16b + 16$.

Hence $0 = b^2 - 14b - 15 = (b - 15)(b + 1)$. Therefore $b = 15$.

2. Preamble for Methods 1, 2, 3

Let $BK = x$ and $PQ = y$.

Since $ABC$ is a right-angled isosceles triangle and $BMLK$ is a square, $CML$ and $AKL$ are also right-angled isosceles triangles. Therefore $AK = CM = x$.

Since $XYZ$ is a right-angled isosceles triangle and $PQRS$ is a square, $XPS$ and $ZQR$ and therefore $YRS$ are also right-angled isosceles triangles. Therefore $XP = ZQ = y$.

Method 1

We have $3y = XZ = AC = AB\sqrt{2} = 2x\sqrt{2}$. So $y = \frac{2\sqrt{2}}{3}x$.

Hence the area of $PQRS = y^2 = \frac{8}{9}x^2 = \frac{8}{9} \times 189 = 168$.

Method 2

We have $2x = AB = \frac{AC}{\sqrt{2}} = \frac{XZ}{\sqrt{2}} = \frac{3y}{\sqrt{2}}$. So $y = \frac{2\sqrt{2}}{3}x$.

Hence the area of $PQRS = y^2 = \frac{8}{9}x^2 = \frac{8}{9} \times 189 = 168$.

Method 3

We have $2x = AB = XY = XS + SY = \sqrt{2}y + \frac{1}{\sqrt{2}}y = (\sqrt{2} + \frac{1}{\sqrt{2}})y = \frac{3\sqrt{2}}{\sqrt{2}}y$. So $y = \frac{2\sqrt{2}}{3}x$.

Hence the area of $PQRS = y^2 = \frac{8}{9}x^2 = \frac{8}{9} \times 189 = 168$. 
Method 4
Joining $B$ to $L$ divides $\triangle ABC$ into 4 congruent right-angled isosceles triangles. Hence the area of $\triangle ABC$ is twice the area of $KLMB$.

Drawing the diagonals of $PQRS$ and the perpendiculars from $P$ to $XS$ and from $Q$ to $RZ$ divides $\triangle XYZ$ into 9 congruent right-angled isosceles triangles.

Hence the area of $PQRS = \frac{4}{9} \times \text{area of } \triangle XYZ = \frac{4}{9} \times \text{area of } \triangle ABC = \frac{4}{9} \times 2 \times \text{area of } KLMB = \frac{8}{9} \times 189 = 168$.

3. Preamble for Methods 1, 2, 3

Since $x$, $y$, and $\frac{x}{y} - \frac{y}{x}$ are all positive, we know that $x > y$. Since $xy = 2048 = 2^{11}$ and $x$ and $y$ are integers, we know that $x$ and $y$ are both powers of 2.

Method 1
Therefore $(x,y) = (2048,1), (1024,2), (512,4), (256,8), (128,16), \text{or (64,32)}$.

Only $(128,16)$ satisfies $\frac{x}{y} - \frac{y}{x} = 7$. $\frac{875}{7} = 7 \frac{7}{8}$, so $x = 128$.

Method 2
Let $x = 2^m$ and $y = 2^n$. Then $m > n$ and $xy = 2^{m+n}$, so $m + n = 11$.

From $\frac{x}{y} - \frac{y}{x} = 7$, we have $2^{m-n} - 2^{n-m} = \frac{63}{8}$.

Let $m - n = t$. Then $2^t - 2^{-t} = \frac{63}{8}$. So $0 = 8(2^t)^2 - 63(2^t) - 8 = (2^t - 8)(8(2^t) + 1)$.

Hence $2^t = 8 = 2^3$, $m - n = 3$, $2m = 14$, and $m = 7$. Therefore $x = 2^7 = 128$.

Method 3
Let $x = 2^m$ and $y = 2^n$. Then $m > n$.

From $\frac{x}{y} - \frac{y}{x} = 7$, we have $x^2 - y^2 = \frac{63}{8}xy = \frac{63}{8} \times 2048 = 63 \times 2^8$.

So $63 \times 2^8 = (x-y)(x+y) = (2^m - 2^n)(2^m + 2^n) = 2^{2n}(2^m - n - 1)(2^n - m + 1)$.

Hence $2^{2n} = 2^8$, $2^{m-n} - 1 = 7$, and $2^{m-n} + 1 = 9$.

Therefore $n = 4$, $2^{m-n} = 8$, and $x = 2^m = 8 \times 2^4 = 2^7 = 128$. 

2
Method 4

Now \( \frac{x}{y} - \frac{y}{x} = 7.875 = 7 \frac{7}{8} \) and \( \frac{x}{y} - \frac{y}{x} = \frac{x^2 - y^2}{xy} = \frac{x^2 - y^2}{2048} \).

So \( x^2 - y^2 = \frac{63}{8} \times 2048 = 63 \times 2^8 = (64 - 1)2^8 = (2^6 - 1)2^8 = 2^{14} - 2^8 \).  

Substituting \( y = 2048 / x = 2^{11} / x \) gives \( x^2 - 2^{22} / x^2 = 2^{14} - 2^8 \).

Hence \( 0 = (x^2)^2 - (2^{14} - 2^8)x^2 - 2^{22} = (x^2 - 2^{14})(x^2 + 2^8) \).

So \( x^2 = 2^{14} \). Since \( x \) is positive, \( x = 2^7 = 128 \).

Method 5

We have \( \frac{x}{y} - \frac{y}{x} = 7.875 = 7 \frac{7}{8} \). Multiplying by \( xy \) gives \( x^2 - y^2 = \frac{63}{8} xy \).

So \( 8x^2 - 63xy - 8y^2 = 0 \) and \( (8x + y)(x - 8y) = 0 \).

Since \( x \) and \( y \) are positive, \( x = 8y, 8y^2 = xy = 2048, y^2 = 256, y = 16, x = 128 \).

Comment. From Method 4 or 5, we don’t need to know that \( x \) and \( y \) are integers to solve this problem.

4. Let the red, blue and green blocks have different weights \( r, b \) and \( g \) kg respectively.

Then we have:

\[
\begin{align*}
5r + 3b + 5g &= w \quad (1) \\
4r + 5b + 4g &= w \quad (2) \\
7r + 4b + ng &= w \quad (3)
\end{align*}
\]

where \( n \) is the number of green blocks.

Subtracting (1) and (2) gives \( 2b = r + g \).

Substituting in (2) gives \( 13b = w \), so \( w \) is a multiple of 13 between 30 and 50.

Hence \( w = 39, b = 3 \) and \( r + g = 6 \).

Method 1

Since \( r + g = 6, r \) is one of the numbers 1, 2, 4, 5.

If \( r \) is 4 or 5, \( 7r + 4b > 39 \) and (3) cannot be satisfied.

If \( r \) is 2, then \( g = 4 \) and (3) gives \( 26 + 4n = 39 \), which cannot be satisfied in integers.

So \( r \) is 1, then \( g = 5 \) and (3) gives \( 19 + 5n = 39 \) and \( n = 4 \).

Hence the total weight in kilograms of 6 red, 7 blue, and 3 green blocks is \( 6 \times 1 + 7 \times 3 + 3 \times 5 = 42 \).

Method 2

Since \( r + g = 6, g \) is one of the numbers 1, 2, 4, 5.

Substituting \( r = 6 - g \) in (3) gives \( (7 - n)g = 15 \). Thus \( g \) is 1 or 5.

If \( g \) is 1, then \( n = -8 \), which is not allowed.

If \( g \) is 5, then \( n = 4 \) and \( r = 1 \).

Hence the total weight in kilograms of 6 red, 7 blue, and 3 green blocks is \( 6 \times 1 + 7 \times 3 + 3 \times 5 = 42 \).
5. Method 1

From symmetry we may assume \( a \leq b \). If \( a = b \), then both are 40 and \( a + b = 80 \). We now assume \( a < b \). As \( a \) increases, \( b \) must decrease to satisfy the equation \( \frac{1}{a} + \frac{1}{b} = \frac{1}{20} \). So \( a < 40 \).

We have \( \frac{1}{b} = \frac{1}{20} - \frac{1}{a} = \frac{a-20}{20a} \). So \( b = \frac{20a}{a-20} \). Since \( a \) and \( b \) are positive, \( a > 20 \).

The table shows all integer values of \( a \) and \( b \).

<table>
<thead>
<tr>
<th>( a )</th>
<th>21</th>
<th>22</th>
<th>24</th>
<th>25</th>
<th>28</th>
<th>30</th>
<th>36</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b )</td>
<td>420</td>
<td>220</td>
<td>120</td>
<td>100</td>
<td>70</td>
<td>60</td>
<td>45</td>
</tr>
</tbody>
</table>

Thus the largest value of \( a + b \) is \( 21 + 420 = 441 \).

Method 2

As in Method 1, we have \( b = \frac{20a}{a-20} \) and \( 20 < a \leq 40 \).

If \( a = 21 \), then \( a + b = 21(1 + 20) = 441 \). If \( a \geq 22 \), then \( a + b \leq 40(1 + 10) = 440 \).

Thus the largest value of \( a + b \) is \( 441 \).

Method 3

We have \( ab = 20(a + b) \). So \((a - 20)(b - 20) = 400 = 2^45^2 \).

Since \( b \) is positive, \( ab > 20a \) and \( a > 20 \). Similarly \( b > 20 \).

From symmetry we may assume \( a \leq b \) hence \( a - 20 \leq b - 20 \).

The table shows all values of \( a - 20 \) and the corresponding values of \( b - 20 \).

<table>
<thead>
<tr>
<th>( a - 20 )</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b - 20 )</td>
<td>400</td>
<td>200</td>
<td>100</td>
<td>50</td>
<td>25</td>
<td>80</td>
<td>40</td>
<td>20</td>
</tr>
</tbody>
</table>

Thus the largest value of \( a + b \) is \( 21 + 420 = 441 \).

Method 4

We have \( ab = 20(a + b) \), so 5 divides \( a \) or \( b \). Since \( b \) is positive, \( ab > 20a \) and \( a > 20 \).

Suppose 5 divides \( a \) and \( b \). From symmetry we may assume \( a \leq b \). The following table gives all values of \( a \) and \( b \).

<table>
<thead>
<tr>
<th>( a )</th>
<th>25</th>
<th>30</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b )</td>
<td>100</td>
<td>60</td>
<td>40</td>
</tr>
</tbody>
</table>

Suppose 5 divides \( a \) but not \( b \). Since \( b(a - 20) = 20a \), 25 divides \( a - 20 \). Let \( a = 20 + 25n \). Then \((20 + 25n)b = 20(20 + 25n + b) \), \( nb = 16 + 20m \), \( n(b - 20) = 16 \). The following table gives all values of \( n \), \( b \), and \( a \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b - 20 )</td>
<td>16</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( b )</td>
<td>36</td>
<td>28</td>
<td>24</td>
<td>22</td>
<td>21</td>
</tr>
<tr>
<td>( a )</td>
<td>45</td>
<td>70</td>
<td>120</td>
<td>220</td>
<td>420</td>
</tr>
</tbody>
</table>

A similar table is obtained if 5 divides \( b \) but not \( a \).

Thus the largest value of \( a + b \) is \( 21 + 420 = 441 \).
Method 5

We have \( ab = 20(a + b) \). So maximising \( a + b \) is equivalent to maximising \( ab \), which is equivalent to minimising \( \frac{1}{ab} \).

Let \( x = \frac{1}{a} \) and \( y = \frac{1}{b} \). We want to minimise \( xy \) subject to \( x + y = \frac{1}{20} \). From symmetry we may assume \( x \geq y \). Hence \( x \geq \frac{1}{410} \).

Thus we want to minimise \( z = x\left(\frac{1}{20} - x\right) \) with \( z > 0 \), hence with \( 0 < x < \frac{1}{20} \). The graph of this function is an inverted parabola with its turning point at \( x = \frac{1}{40} \). So the minimum occurs at \( x = \frac{1}{21} \). This corresponds to \( y = \frac{1}{20} - \frac{1}{21} = \frac{1}{420} \).

Thus the largest value of \( a + b \) is \( 21 + 420 = 441 \).

6. Method 1

Let \( b \) be the number of black socks and \( w \) the number of white ones. If \( b \) or \( w \) is 0, then the probability of withdrawing a pair of socks of the same colour would be 1. So \( b \) and \( w \) are positive. From symmetry we may assume that \( b \geq w \).

The number of pairs of black socks is \( b(b-1)/2 \). The number of pairs of white socks is \( w(w-1)/2 \). The number of pairs of socks with one black and the other white is \( bw \).

The probability of selecting a pair of socks of the same colour is the same as the probability of selecting a pair of socks of different colour. Hence \( b(b-1)/2 + w(w-1)/2 = bw \) or
\[
b(b-1) + w(w-1) = 2bw
\]

Let \( d = b - w \). Then \( w = b - d \) and
\[
\begin{align*}
b(b-1) + (b-d)(b-d-1) & = 2b(b-d) \\
b^2 - b + b^2 - bd - b - bd + d^2 + d & = 2b^2 - 2bd \\
-2b + d^2 + d & = 0 \\
d(d+1) & = 2b
\end{align*}
\]

The following table shows all possible values of \( d \). Note that \( b + w = 2b - d = d^2 \).

<table>
<thead>
<tr>
<th>( d )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>( \geq 8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b )</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>15</td>
<td>21</td>
<td>28</td>
<td></td>
</tr>
<tr>
<td>( b + w )</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
<td>36</td>
<td>49</td>
<td>( \geq 64 )</td>
</tr>
</tbody>
</table>

Thus the largest value of \( b \) is \( 28 \).
Preamble for Methods 2, 3, 4

Let $b$ be the number of black socks and $w$ the number of white ones. If $b$ or $w$ is 0, then the probability of withdrawing a pair of socks of the same colour would be 1. So $b$ and $w$ are positive. From symmetry we may assume that $b \geq w$.

The pair of socks that Justin withdraws are either the same colour or different colours. So the probability that he draws a pair of socks of different colours is $1 - \frac{0}{b+w-1}$.

The following diagram shows the probabilities of withdrawing one sock at a time.

\[
\begin{array}{c}
\text{B} \\
\text{W} \\
\text{W} \\
\text{B} \\
\text{B} \\
\text{W} \\
\text{W}
\end{array}
\]

So the probability that Justin draws a pair of socks of different colours is $\frac{2bw}{(b+w)(b+w-1)}$.

Hence $4bw = b^2 + 2bw + w^2 - b - w$ and $b^2 - 2bw + w^2 - b - w = 0$.

Method 2

We have $b^2 - (2w+1)b + (w^2 - w) = 0$.

The quadratic formula gives $b = \frac{2w+1 \pm \sqrt{(2w+1)^2 - 4(w^2 - w)}}{2} = \frac{2w+1 \pm \sqrt{8w+1}}{2}$.

If $b = \frac{2w+1 - \sqrt{8w+1}}{2} = w + \frac{1}{2} - \frac{1}{2} \sqrt{8w+1}$, then $b \leq w + \frac{1}{2} - \frac{1}{2} \sqrt{9} = w - 1 < w$.

So $b = \frac{2w+1 + \sqrt{8w+1}}{2}$.

Now $w < 25$ otherwise $b + w \geq 2w \geq 50$.

Since $b$ increases with $w$, we want the largest value of $w$ for which $8w + 1$ is square. Thus $w = 21$ and the largest value of $b$ is $\frac{2w+1 + \sqrt{169}}{2} = (43 + 13)/2 = 28$.

Method 3

We have $b + w = (b - w)^2$. Thus $b + w$ is a square number less than 50 and greater than 1.

The following tables gives all values of $b + w$ and the corresponding values of $b - w$ and $b$.

<table>
<thead>
<tr>
<th>$b + w$</th>
<th>4</th>
<th>9</th>
<th>16</th>
<th>25</th>
<th>36</th>
<th>49</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b - w$</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>$2b$</td>
<td>6</td>
<td>12</td>
<td>20</td>
<td>30</td>
<td>42</td>
<td>56</td>
</tr>
<tr>
<td>$b$</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>15</td>
<td>21</td>
<td>28</td>
</tr>
</tbody>
</table>

Thus the largest value of $b$ is 28.
**Method 4**

We have \( b + w = (b - w)^2 \). Also \( w < 25 \) otherwise \( b + w \geq 2w \geq 50 \). For a fixed value of \( w \), consider the line \( y = w + b \) and parabola \( y = (b - w)^2 \). These intersect at a unique point for \( b \geq w \). For each value of \( w \) we guess and check a value of \( b \) for which the line and parabola intersect.

<table>
<thead>
<tr>
<th>( w )</th>
<th>( b )</th>
<th>( b + w )</th>
<th>( (b - w)^2 )</th>
<th>( b + w = (b - w)^2 ? )</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>31</td>
<td>55</td>
<td>49</td>
<td>( b + w &lt; (b - w)^2 )</td>
</tr>
<tr>
<td>32</td>
<td>66</td>
<td>64</td>
<td>( b + w &lt; (b - w)^2 )</td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>30</td>
<td>53</td>
<td>49</td>
<td>( b + w &gt; (b - w)^2 )</td>
</tr>
<tr>
<td>31</td>
<td>54</td>
<td>64</td>
<td>( b + w &lt; (b - w)^2 )</td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>29</td>
<td>51</td>
<td>49</td>
<td>( b + w &gt; (b - w)^2 )</td>
</tr>
<tr>
<td>30</td>
<td>52</td>
<td>64</td>
<td>( b + w &lt; (b - w)^2 )</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>27</td>
<td>48</td>
<td>36</td>
<td>( b + w &gt; (b - w)^2 )</td>
</tr>
<tr>
<td>28</td>
<td>49</td>
<td>49</td>
<td>( b + w = (b - w)^2 )</td>
<td></td>
</tr>
</tbody>
</table>

As \( w \) decreases, the line \( y = w + b \) shifts down and the parabola \( y = (b - w)^2 \) shifts left so their point of intersection shifts left. So \( b \) decreases as \( w \) decreases. Thus the largest value of \( b \) is 28.

**Comment.** We have \( b + w = (b - w)^2 \). Let \( b - w = n \). Then \( b + w = n^2 \). Hence \( b = (n^2 + n)/2 = n(n + 1)/2 \) and \( w = (n^2 - n)/2 = (n - 1)n/2 \). Thus \( w \) and \( b \) are consecutive triangular numbers.

7. **Method 1**

Let \( c_n \) be the number of codes that have exactly \( n \) digits.

For \( n \geq 4 \), a code with \( n \) digits ends with 1 or 10 or 100.

If the code ends in 1, then the string that remains when the end digit is removed is also a code. So the number of codes that end in 1 and have exactly \( n \) digits equals \( c_{n-1} \).

If the code ends in 10, then the string that remains when the last 2 digits are removed is also a code. So the number of codes that end in 10 and have exactly \( n \) digits equals \( c_{n-2} \).

If the code ends in 100, then the string that remains when the last 3 digits are removed is also a code. So the number of codes that end in 100 and have exactly \( n \) digits equals \( c_{n-3} \).

Hence, for \( n \geq 4 \), \( c_n = c_{n-1} + c_{n-2} + c_{n-3} \).

By direct counting, \( c_1 = 2, \ c_2 = 4, \ c_3 = 7 \). The table shows \( c_n \) for \( 1 \leq n \leq 11 \).

<table>
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<tr>
<th>( n )</th>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<tbody>
<tr>
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<td>81</td>
<td>149</td>
<td>274</td>
<td>504</td>
<td>927</td>
</tr>
</tbody>
</table>

Thus the number of codes that have exactly 11 digits is 927.
Method 2

Let \( c_n \) be the number of codes that have exactly \( n \) digits.

A code ends with 0 or 1.

Suppose \( n \geq 5 \). If a code ends with 1, then the string that remains when the end digit is removed is also a code. So the number of codes that end with 1 and have exactly \( n \) digits equals \( c_{n-1} \).

If a code with \( n \) digits ends in 0, then the string that remains when the end digit is removed is a code with \( n-1 \) digits that does not end with two 0s. If a code with \( n-1 \) digits ends with two 0s, then it ends with 100. If the 100 is removed then the string that remains is an unrestricted code that has exactly \( n-4 \) digits. So the number of codes with \( n-1 \) digits that do not end with two 0s is \( c_{n-1} - c_{n-4} \).

Hence, for \( n \geq 5 \),
\[
 c_n = 2c_{n-1} - c_{n-4}.
\]

By direct counting, \( c_1 = 2, c_2 = 4, c_3 = 7, c_4 = 13 \). The table shows \( c_n \) for \( 1 \leq n \leq 11 \).

<table>
<thead>
<tr>
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<tr>
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<td>149</td>
<td>274</td>
<td>504</td>
<td>927</td>
</tr>
</tbody>
</table>

Thus the number of codes that have exactly 11 digits is 927.

Comment. The equation \( c_n = 2c_{n-1} - c_{n-4} \) can also be derived from the equation
\[
 c_n = c_{n-1} + c_{n-2} + c_{n-3}.
\]

For \( n \geq 5 \) we have \( c_{n-1} = c_{n-2} + c_{n-3} + c_{n-4} \).

Hence \( c_n = c_{n-1} + (c_{n-1} - c_{n-4}) = 2c_{n-1} - c_{n-4} \).

8. Method 1

The square of \( n \) has the same last three digits of \( n \) if and only if \( n^2 - n = n(n-1) \) is divisible by 1000 = \( 2^3 \times 5^3 \).

As \( n \) and \( n-1 \) are relatively prime, only one of those two numbers is even and only one of them can be divisible by 5. This yields the following cases.

Case 1. \( n \) is divisible by both \( 2^3 \) and \( 5^3 \). Then \( n \geq 1000 \), a contradiction.

Case 2. \( n-1 \) is divisible by both \( 2^3 \) and \( 5^3 \). Then \( n \geq 1001 \), a contradiction.

Case 3. \( n \) is divisible by \( 2^3 \) and \( n-1 \) is divisible by \( 5^3 \). The second condition implies that \( n \) is one of the numbers 1, 126, 251, 376, 501, 626, 751, 876. Since \( n \) is also divisible by 8, this leaves \( n = 376 \).

Case 4. \( n \) is divisible by \( 5^3 \) and \( n-1 \) is divisible by \( 2^3 \). The first condition implies that \( n \) is one of the numbers 125, 250, 375, 500, 625, 750, 875. But \( n \) must also leave remainder 1 when divided by 8, which leaves \( n = 625 \).

Therefore \( n = 625 \).
Method 2

We want a number \( n \) and its square to have the same last three digits.

First, \( n \) and \( n^2 \) should have the same last digit. Squaring each of the digits from 0 to 9 shows that the last digit of \( n \) must be 0, 1, 5 or 6.

Second, \( n \) and \( n^2 \) should have the same last two digits. Squaring each of the 2-digit numbers 00 to 90, 01 to 91, 05 to 95, and 06 to 96 as in the following table shows that the last two digits of \( n \) must be 00, 01, 25 or 76.

\[
\begin{array}{cccccc}
 n^2 & n & n^2 & n & n^2 & n \\
00 & 00 & 01 & 01 & 05 & 25 \\
10 & 00 & 11 & 21 & 15 & 25 \\
20 & 00 & 21 & 41 & 25 & 26 \\
30 & 00 & 31 & 61 & 35 & 25 \\
40 & 00 & 41 & 81 & 45 & 25 \\
50 & 00 & 51 & 01 & 55 & 25 \\
60 & 00 & 61 & 21 & 65 & 25 \\
70 & 00 & 71 & 41 & 75 & 25 \\
80 & 00 & 81 & 61 & 85 & 25 \\
90 & 00 & 91 & 81 & 95 & 25 \\
\end{array}
\]

Finally, \( n \) and \( n^2 \) should have the same last three digits. Squaring each of the 3-digit numbers 000 to 900, 001 to 901, 025 to 925, and 076 to 976 as in the following table shows that the last three digits of \( n \) must be 000, 001, 625 or 376.

\[
\begin{array}{cccccc}
 n^2 & n & n^2 & n & n^2 & n \\
000 & 000 & 001 & 001 & 025 & 625 \\
100 & 000 & 101 & 201 & 125 & 625 \\
200 & 000 & 201 & 401 & 225 & 625 \\
300 & 000 & 301 & 601 & 325 & 625 \\
400 & 000 & 401 & 801 & 425 & 625 \\
500 & 000 & 501 & 001 & 525 & 625 \\
600 & 000 & 601 & 201 & 625 & 625 \\
700 & 000 & 701 & 401 & 725 & 625 \\
800 & 000 & 801 & 601 & 825 & 625 \\
900 & 000 & 901 & 801 & 925 & 625 \\
\end{array}
\]

Therefore \( n = 625 \).
9. **Preamble**

Since $ABCD$ is a cyclic quadrilateral, $\angle DCA = \angle DBA$. Since $AB \parallel CD$, $\angle DCA = \angle CAB$.

So $\triangle AMB$ is isosceles. Similarly $\triangle CMD$ is isosceles.

Extend $MO$ to intersect $AB$ at $X$ and $CD$ at $Y$.

Since $OA = OB$, triangles $AMO$ and $BMO$ are congruent. So $\angle AMO = \angle BMO$. Since $\angle AMD = 60^\circ$, $\angle AMB = 120^\circ$ and $\angle AMO = \angle BMO = 60^\circ$. Hence triangles $AMX$ and $BMX$ are congruent and have angles $30^\circ$, $60^\circ$, $90^\circ$. Similarly $DMY$ and $CMY$ are congruent $30$-$60$-$90$ triangles.

---

**Method 1**

We know that $X$ and $Y$ are the midpoints of $AB$ and $CD$ respectively. Let $2x$ and $2y$ be the lengths of $AB$ and $CD$ respectively. From the $30$-$90$-$60$ triangles $AXM$ and $CYM$ we have $XM = \frac{x}{\sqrt{3}}$ and $YM = \frac{y}{\sqrt{3}}$.

From the right-angled triangles $AXO$ and $CYO$, Pythagoras gives

$$AO^2 = x^2 + \left(\frac{x}{\sqrt{3}} - 10\right)^2 = \frac{4}{3}x^2 + 100 - \frac{20}{\sqrt{3}}x$$
$$CO^2 = y^2 + \left(\frac{y}{\sqrt{3}} + 10\right)^2 = \frac{4}{3}y^2 + 100 + \frac{20}{\sqrt{3}}y$$

These equations also hold if $O$ lies outside the trapezium $ABCD$.

Since $AO = CO$, we have $\frac{4}{3}(x^2 - y^2) = \frac{20}{\sqrt{3}}(x + y)$, $x^2 - y^2 = 5\sqrt{3}(x + y)$, $x - y = 5\sqrt{3}$ and $AB - CD = 2(x - y) = 10\sqrt{3}$. 

---
Method 2

We know that $\angle ABD = 30^\circ$. Since O is the centre of the circle we have $\angle AOD = 2\angle ABD = 60^\circ$. Thus $\angle AOD = \angle AMD$, hence $AOMD$ is cyclic. Since $OA = OD$ and $\angle AOD = 60^\circ$, $\triangle AOD$ is equilateral.

Rotate $\triangle AOM$ $60^\circ$ anticlockwise about $A$ to form triangle $ADN$.

Since $AOMD$ is cyclic, $\angle AOM + \angle ADM = 180^\circ$. Hence $MDN$ is a straight line. Since $\angle AMD = 60^\circ$ and $AM = AN$, $\triangle AMN$ is equilateral. So $AM = MN = MD + DN = MD + MO$.

[Alternatively, applying Ptolemy’s theorem to the cyclic quadrilateral $AOMD$ gives $AO \times MD + AD \times MO = AM \times OD$. Since $AO = AD = OD$, cancelling these gives $MD + MO = AM$.]

We know that $X$ and $Y$ are the midpoints of $AB$ and $CD$ respectively. From the 30-90-60 triangles $AXM$ and $DYM$ we have $AX = \frac{\sqrt{3}}{2} AM$ and $DY = \frac{\sqrt{3}}{2} DM$.

So $AB - CD = 2(\frac{\sqrt{3}}{2} AM - \frac{\sqrt{3}}{2} DM) = \sqrt{3} MO = 10\sqrt{3}$.  

1
Method 3

As in Method 2, \( \triangle AOD \) is equilateral.

Let \( P \) and \( Q \) be points on \( AB \) and \( BD \) respectively so that \( DP \perp AB \) and \( OQ \perp BD \).

From the 30-60-90 triangle \( BDP \), \( DP = \frac{1}{2}BD \). Since \( OB = OD \), triangles \( DOQ \) and \( BOQ \) are congruent. Hence \( DQ = \frac{1}{2}DB = DP \). So triangles \( APD \) and \( OQD \) are congruent. Therefore \( AP = OQ \).

From the 30-60-90 triangle \( OMQ \), \( OQ = \frac{\sqrt{3}}{2}OM = 5\sqrt{3} \).

So \( AB - CD = 2AX - 2DY = 2AX - 2PX = 2AP = 10\sqrt{3} \).
Method 4

Let \( x = BM \) and \( y = DM \). From the 30-90-60 triangles \( BXM \) and \( DYM \) we have \( BX = \frac{\sqrt{3}}{2}x \) and \( DY = \frac{\sqrt{3}}{2}y \). Since \( X \) and \( Y \) are the midpoints of \( AB \) and \( CD \) respectively, \( AB - CD = \sqrt{3}(x - y) \).

Let \( Q \) be the point on \( BD \) so that \( OQ \perp BD \).

Since \( BO = DO \), triangles \( BQO \) and \( DQO \) are congruent and \( BQ = DQ \). Therefore \( BQ = (x + y)/2 \) and \( MQ = x - BQ = (x - y)/2 \).

Since \( BXM \) is a 30-90-60 triangle, \( \triangle OQM \) is also 30-90-60. Therefore \( MQ = \frac{1}{2}MO = 5 \). So \( AB - CD = 2\sqrt{3}MQ = 10\sqrt{3} \).
Method 5

We know that triangles $AMB$ and $DMC$ have the same angles. Let the line that passes through $O$ and is parallel to $AC$ intersect $AB$ at $Q$ and $BD$ at $P$. Then $\angle BQP = \angle BAM$ and $\angle BPQ = \angle BMA$. So triangles $BPQ$ and $CMD$ are similar.

Now $\angle QPD = \angle AMD = 60^\circ$. So $\triangle OMP$ is equilateral. Let the line that passes through $O$ and is perpendicular to $BD$ intersect $BD$ at $R$. Thus $R$ bisects $PM$. Since $OD = OB$, triangles $OBR$ and $ODR$ are congruent and $R$ bisects $BD$. Hence $DM = BP$ and triangles $BPQ$ and $CMD$ are congruent. So $AB - CD = AQ$.

Draw $QN$ parallel to $OM$ with $N$ on $AM$. Then $QN = OM = 10$ and $QN \perp AB$. So $\triangle ANQ$ is 30-60-90. Hence $AN = 20$ and, by Pythagoras, $AB - CD = \sqrt{400 - 100} = \sqrt{300} = 10\sqrt{3}$.

Comment. Notice that $AB - CD$ is independent of the radius of the circumcircle $ABCD$. This is true for all cyclic trapeziums. If $\angle AMD = \alpha$, then by similar arguments to those above we can show that $AB - CD = 2MO\sin\alpha$. 

14
10. (a) Mark cells of the grid by coordinates, with (1, 1) being the cell in the lower-left corner of the grid. There are many ways of achieving a covering with exactly \( n \) black cells visible. Here’s three.

**Method 1**

Putting each new tile *above* all previous tiles it overlaps with, place tiles in the following order with their lower-left cells on the listed grid cells:

- (1, 1),
- (1, 2), (2, 1),
- (1, 3), (2, 2), (3, 1),
- (1, 4), (2, 3), (3, 2), (4, 1),
- and so on.

Continue this procedure to give black cells on the ‘diagonal’ just below the main diagonal and only white cells below. The following diagram shows this procedure for \( n = 5 \).

![Diagram showing method 1 for \( n = 5 \)](image)

Start then in the upper-right corner and create black cells on the ‘diagonal’ just above the main diagonal and only white cells above. Finally put \( n - 1 \) tiles along the main diagonal. That will give \( n \) black cells on the main diagonal and white cells everywhere else.
Method 2

Rotate all tiles so that the lower-left and upper-right cells are black. Putting each new tile \textit{underneath} all previous tiles it overlaps with, place tiles in the following order with their lower-left cells on the listed grid cells:

- (1, 1),
- (2, 2), (1, 2), (2, 1),
- (3, 3), (2, 3), (1, 3), (3, 2), (3, 1),
- (4, 4), (3, 4), (2, 4), (1, 4), (4, 3), (4, 2), (4, 1),

and so on.

Continuing this procedure gives $n$ black cells on the diagonal and white cells everywhere else. The following diagram shows this procedure for $n = 5$. 

![Diagram showing the procedure for Method 2 with $n = 5$.]
Method 3

Putting each new tile above all previous tiles it overlaps with, place tiles in the following order with their lower-left cells on the listed grid cells:

\[(1, 1), (2, 1), (3, 1), \ldots, (n - 1, 1),\]
\[(1, 2), (1, 3), \ldots, (1, n - 1),\]
\[(n - 1, n - 1), (n - 2, n - 1), \ldots, (1, n - 1),\]
\[(n - 1, n - 2), (n - 1, n - 3), \ldots, (n - 1, 1),\]

The following diagram shows this procedure for \(n = 5\).

This gives a single border of all white cells except for black cells in the top-left and bottom-right corners of the grid. Now repeat this procedure for the inner \((n - 2) \times (n - 2)\) grid, then the inner \((n - 4) \times (n - 4)\) grid, and so on until an inner \(1 \times 1\) or \(2 \times 2\) grid remains. In both cases a single tile can cover the remaining uncovered grid cell(s) to produce a total covering that has \(n\) black cells on the diagonal and white cells everywhere else.

(b) Suppose there is a covering of the \(n \times n\) grid that has less than \(n\) black cells visible. Then there must be a row in which all visible cells are white. Any tile that overlaps this row has exactly two cells that coincide with cells in the row. These two cells are in the same row of the tile so one is white and one is black. Call these two cells a half-tile. Consider all half-tiles that cover cells in the row. Remove any half-tiles that have neither cell visible. The remaining half-tiles cover the row and all their visible cells are white.

Consider any half-tile \(H_1\). The black cell of \(H_1\) must be covered by some half-tile \(H_2\) and the white cell of \(H_1\) must be visible. The black cell of \(H_2\) must be covered by some half-tile \(H_3\) and the white cell of \(H_2\) must be visible. Thus we have a total of two visible white cells in the row. The black cell of \(H_3\) must be covered by some half-tile \(H_4\) and the white cell of \(H_3\) must be visible. Thus we have a total of three visible white cells in the row.

So we may continue until we have a half-tile \(H_{n-1}\) plus a total of \(n - 2\) visible white cells in the row. The black cell of \(H_{n-1}\) must be covered by some half-tile \(H_n\) and the white cell of \(H_{n-1}\) must be visible. Thus we have a total of \(n - 1\) visible white cells in the row. As there are only \(n\) cells in the row, \(H_n\) must cover one of the visible white cells. This is a contradiction. So every covering of the \(n \times n\) grid has at least \(n\) black cells visible.

(c) From (a) and (b), the minimum number of visible black cells is \(n\). From symmetry, the minimum number of visible white cells is \(n\). Hence the maximum number of visible black cells is \(n^2 - n\).
Investigation

(i) If a covering of a $3 \times 3$ grid has exactly 3 visible black cells, then the argument in Part (b) above shows that each row and each column must have exactly one visible black cell. The following diagram shows all possible patterns with exactly 3 black cells.

From symmetry we only need to consider the first two patterns. A covering to achieve the first pattern was given in Part (a) above. The second pattern can be achieved from the first by rotating a tile $90^\circ$ and placing it in the bottom-right corner of the grid.

(ii) The last tile to be placed shows two visible black cells and they share a vertex. However, in the following pattern no two black cells share a vertex.

Thus not all patterns of 4 black cells and 12 white cells on a $4 \times 4$ grid can be achieved by a covering in which each new tile is placed on top of all previous tiles that it overlaps.

Comment. This pattern can be achieved however if new tiles may be placed under previous tiles.

(iii) By the same argument as that in Part (b) above, the number of black cells exposed in any covering of the $n \times m$ grid is at least $m$.

We now show $m$ is achievable. Number the columns 1 to $m$. Using the procedure in Part (a) Method 1 above, cover columns 1 to $n$ to give $n$ black cells on the main diagonal and white cells everywhere else. Now apply the same covering on columns 2 to $n + 1$, then on columns 3 to $n + 2$, and so on, finishing with columns $m - n + 1$ to $m$. This procedure covers the entire $n \times m$ grid leaving exactly $m$ black cells visible. The following diagram shows this procedure for $n = 3$.

So the minimum number of visible black cells in any covering of the $n \times m$ grid is $m$. From symmetry, the minimum number of visible white cells in any covering of the $n \times m$ grid is $m$. Hence the maximum number of visible black cells in any covering of the $n \times m$ grid is $nm - m = m(n - 1)$. 

bonus 1

bonus 2
Marking Scheme

1. A correct approach.  
   Correct answer (15).  

2. A correct approach.  
   Correct answer (168).  

3. A correct approach.  
   Substantial progress.  
   Correct answer (128).  

4. Three correct equations.  
   Correct value of \( w \).  
   Correct answer (42).  

5. A correct approach.  
   Further progress.  
   Substantial progress.  
   Correct answer (441).  

6. A correct approach.  
   A useful equation.  
   Substantial progress.  
   Correct answer (28).  

7. A correct approach.  
   Further progress.  
   Substantial progress.  
   Correct answer (927).  

8. A correct approach.  
   Further progress.  
   Substantial progress.  
   Correct answer (625).  

9. Establishing triangles \( AMB \) and \( CMD \) are isosceles.  
   Establishing angles \( AMO \) and \( BMO \) are 60°.  
   Further progress.  
   Substantial progress.  
   Correct answer (10\( \sqrt{3} \)).  

10. (a) A correct covering with exactly \( n \) black cells visible.  
    (b) Correct proof that at least \( n \) black cells will be visible.  
    (c) Correct answer \( (n^2 - n) \) and proof.  

   **Investigation:**  
   (i) Correct coverings for all patterns.  
   (ii) A convincing explanation.  
   (iii) Correct answer \( (m(n - 1)) \) and proof.  

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## AUSTRALIAN INTERMEDIATE MATHEMATICS OLYMPIAD STATISTICS

### DISTRIBUTION OF AWARDS/SCHOOL YEAR

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<th>YEAR</th>
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ĐỀ THI VÀ ĐÁP ÁN CHI TIẾT

CUỘC THI VÔ ĐỊCH THÔNG KÊ TRUNG CẤP ÚC MÔNG NĂM 2013
Australian Intermediate Mathematics Olympiad 2013
Questions

1. Find the area in cm$^2$ of a rhombus whose side length is 29 cm and whose diagonals differ in length by 2 cm.

   [2 marks]

2. How many 4-digit numbers are there whose digit product is 60?

   [2 marks]

3. A base 7 three-digit number has its digits reversed when written in base 9. Find the decimal representation of the number.

   [3 marks]

4. The prime numbers $p$, $q$, $r$ satisfy the simultaneous equations $pq + pr = 80$ and $pq + qr = 425$. Find the value of $p + q + r$.

   [3 marks]

5. How many pairs of 3-digit palindromes are there such that when they are added together, the result is a 4-digit palindrome? For example, $232 + 989 = 1221$ gives one such pair.

   [4 marks]

6. $ABC$ is an equilateral triangle with side length $2013\sqrt{3}$. Find the largest diameter for a circle in one of the regions between $\triangle ABC$ and its inscribed circle.

   [4 marks]

7. If $a$, $b$, $c$, $d$ are positive integers with sum 63, what is the maximum value of $ab + bc + cd$?

   [4 marks]

PLEASE TURN OVER THE PAGE FOR QUESTIONS 8, 9 AND 10
8. A circle meets the sides of an equilateral triangle $ABC$ at six points $D, E, F, G, H, I$ as shown. If $AE = 4$, $ED = 26$, $DC = 2$, $FG = 14$, and the circle with diameter $HI$ has area $\pi b$, find $b$.

9. A box contains some identical tennis balls. The ratio of the total volume of the tennis balls to the volume of empty space surrounding them in the box is $1 : k$, where $k$ is an integer greater than 1.

A prime number of tennis balls is removed from the box. The ratio of the total volume of the remaining tennis balls to the volume of empty space surrounding them in the box is $1 : k^2$. Find the number of tennis balls that were originally in the box.

10. I have a $1 \times 1$ m square, which I want to cover with three circular discs of equal size (which are allowed to overlap). Show that this is possible if the discs have diameter 1008 mm.

Investigation

Two discs of equal diameter cover a $1 \times 1$ m square. Find their minimum diameter.
1. **Method 1**

Let the diagonals be $2x$ and $2x + 2$. The diagonals of a rhombus bisect each other at right angles. Hence they partition the rhombus into four congruent right-angled triangles each with hypotenuse 29. Thus the area of the rhombus is $4 \times \frac{1}{2}x(x + 1) = 2x^2 + 2x$.

From Pythagoras, $x^2 + (x + 1)^2 = 29^2 = 841$. So $2x^2 + 2x = 840$.

**Method 2**

The diagonals of a rhombus bisect each other at right angles. Hence they partition the rhombus into four congruent right-angled triangles each with hypotenuse 29 and short sides differing by one.

Because the small angles of these triangles are complementary, the triangles can be rearranged to form a square of side 29 with a unit square hole.

Thus the area of the rhombus is $29^2 - 1 = 840$.

2. Since $60 = 2^2 \times 3 \times 5$, only the digits 1, 2, 3, 4, 5 and 6 can be used. The only combinations of four of these digits whose product is 60 are (1, 2, 5, 6), (1, 3, 4, 5), (2, 2, 3, 5).

There are 24 ways to arrange four different digits and 12 ways to arrange four digits of which two are the same. So the total number of required 4-digit numbers is $24 + 24 + 12 = 60$.

3. Call the number $(abc)_7$, where $a$, $b$, $c$ are digits less than 7. Then $(abc)_7 = (cba)_9$.

Hence $49a + 7b + c = 81c + 9b + a$.

So $48a = 80c + 2b$ or $24a = 40c + b$.

Then $b = 24a - 40c = 8(3a - 5c)$.

Since $b$ is both a digit less than 7 and a multiple of 8, it must be 0.

Now we have $3a - 5c = 0$, or $3a = 5c$.

Since $a$ and $c$ are digits less than 7, $a = 5$ and $c = 3$.

So the number is $(503)_7 = 5 \times 49 + 3 \times 1 = 248$. 
4. Method 1

We have \( p(q + r) = 80 = 2^4 \times 5, \) \( q(p + r) = 425 = 5^2 \times 17, \) \( r(q - p) = 345 = 3 \times 5 \times 23. \)

So \( p = 2 \) or \( 5, \) \( q = 5 \) or \( 17, \) \( r = 3, 5, \) or \( 23. \)

From \( p(q + r) = 80, \) if \( p = 5 \) then \( q + r = 16, \) which has no solution.

So \( p = 2, \) then \( q + r = 40, \) hence \( q = 17 \) and \( r = 23. \)

Therefore \( p + q + r \) is \( 2 + 17 + 23 = 42. \)

Method 2

We have \( p(q + r) = 80 = 2^4 \times 5 \) and \( q(p + r) = 425 = 5^2 \times 17. \)

So \( p = 2 \) or \( 5 \) and \( q = 5 \) or \( 17. \)

From \( q(p + r) = 425, \) if \( q = 5 \) then \( p + r = 85. \)

So \( r = 83 \) or \( 80, \) which both contradict \( p(q + r) = 80. \)

Hence \( q = 17 \) and \( p + r = 25. \) Since \( r \) is prime, \( p = 2 \) and \( r = 23. \)

Therefore \( p + q + r \) is \( 2 + 17 + 23 = 42. \)

5. Write the sum as follows:

\[
\begin{align*}
aba \\
cdc \\
efe
\end{align*}
\]

It is clear from the 1000s column that \( e = 1. \)

So from the units column, \( a + c = 1 \) or \( a + c = 11. \)

But the carry from the 100s column means that \( a + c = 11. \)

Method 1

Without loss of generality, take \( a < c. \)

So the possible solution pairs for \((a, c)\) are: \((2, 9), (3, 8), (4, 7), (5, 6).\)

Now the 10s column gives \( b + d + 1 = f \) or \( b + d + 1 = f + 10. \)

Case 1: \( b + d + 1 = f. \)

Since there is no carry to the 100s column and \( a + c = 11, \) we have \( f = 1. \)

Thus \( b + d = 0, \) hence \( b = d = 0. \)

So this gives four solutions:

\[
202 + 909 = 1111, \ 303 + 808 = 1111, \ 404 + 707 = 1111, \ 505 + 606 = 1111.
\]

Case 2: \( b + d + 1 = f + 10. \)

Since there is a carry to the 100s column, \( a + c + 1 = 12. \)

Then \( f = 2, \) hence \( b + d = 11. \)

For each pair of values of \( a \) and \( c, \) there are then eight solution pairs for \((b, d): (2, 9), (3, 8), (4, 7), (5, 6), (6, 5), (7, 4), (8, 3), (9, 2).\)

So this gives \( 4 \times 8 = 32 \) solutions \((222 + 999 = 1221, 232 + 989 = 1221, \text{etc}).\)

Hence the number of solution pairs overall is \( 4 + 32 = 36. \)

Method 2

We know that \( e = 1, \) \( a + c = 11, \) and the carry from the 10s column is at most 1.

Hence \( f = 1 \) or \( 2. \) Therefore \( b + d = 0 \) or \( 11 \) respectively.

For each pair of values of \( a \) and \( c, \) there are then nine solution pairs for \((b, d): (0, 0), (2, 9), (3, 8), (4, 7), (5, 6), (6, 5), (7, 4), (8, 3), (9, 2).\)

Thus the number of required pairs of 3-digit palindromes is \( 4 \times 9 = 36. \)
6. \textbf{Method 1}

Let $I$ be the incircle of $\triangle ABC$ and let $J$ be the largest circle in the top region between $\triangle ABC$ and $I$.

Let $R$ be the incentre of $\triangle ABC$. Then $AR$ bisects $\angle BAC$. Extend $AR$ to meet $BC$ at $X$. Since $\triangle ABC$ is equilateral, $X$ is the midpoint of $BC$. By symmetry, $R$ lies on all medians of $\triangle ABC$. Hence $RX = \frac{1}{3}AX$. $AX$ is also perpendicular to $BC$. Since $J$ touches $AB$ and $AC$, its centre is also on $AX$. Hence $I$ and $J$ touch at some point $Y$ on $AX$. Let their common tangent meet $AB$ at $P$ and $AC$ at $Q$. Then $PQ$ and $BC$ are parallel. Hence $\triangle APQ$ is similar to $\triangle ABC$.

So $\triangle APQ$ is equilateral and its altitude $AY = AX - YX = AX - \frac{2}{3}AX = \frac{1}{3}AX$. Since $J$ is the incircle of $\triangle APQ$, its radius is $\frac{1}{2}AY = \frac{1}{6}AX$. Since $\triangle ABX$ is 30-60-90, $AX = \sqrt{3} \times 1006.5\sqrt{3} = 3 \times 1006.5$. So the diameter of $J = 2 \times \frac{1}{6}AX = 2 \times 1006.5/3 = 671$. 

\[ \text{Diagram} \]
Method 2

Let \( r \) be the radius of the smaller circle.

Let \( P \) be the incentre of \( \triangle ABC \). Then \( AP \) bisects \( \angle BAC \) and \( CP \) bisects \( \angle ACB \). Extend \( AP \) to meet \( BC \) at \( X \). Since \( \triangle ABC \) is equilateral, \( X \) is the midpoint of \( BC \), \( AX \) is perpendicular to \( BC \), and \( PA = PC \).

Thus \( \triangle ABX \) and \( \triangle CPX \) are 30-60-90. Hence \( AX = \sqrt{3} \times 1006.5 \sqrt{3} = 3 \times 1006.5 \) and \( PX = XC/\sqrt{3} = 1006.5 \). So \( AP = 2PX \).

Since the smaller circle touches \( AC \) and \( BC \), its centre, \( Q \), lies on \( CP \). Let \( Y \) be the point where the smaller circle touches \( BC \). Then \( QY \) is perpendicular to \( BC \). Hence \( \triangle CYQ \) is similar to \( \triangle CXP \).

So \( \frac{r}{XP} = \frac{CQ}{CP} = \frac{CQ}{AP} = \frac{CQ}{2PX} \).

Hence \( 2r = CQ = CP - QP = AP - QP = 2 \times 1006.5 - (1006.5 + r) \).
Therefore \( 3r = 1006.5 \) and \( 2r = 2013/3 = 671 \).
7. Method 1

For all real numbers x and y, we have \((x - y)^2 \geq 0\). So \(x^2 + y^2 \geq 2xy\), hence \((x + y)^2 \geq 4xy\) and \(xy \leq (x + y)^2 / 4\).

Letting \(x = a + c\) and \(y = b + d\) gives \((a + c)(b + d) \leq (a + b + c + d)^2 / 4\). So \(ab + bc + cd + da \leq 63^2 / 4 = 3969 / 4 = 992.25\).

Since \(a, b, c, d\) are positive integers, the last inequality can be written as \(ab + bc + cd + da \leq 992\). Hence \(ab + bc + cd + da \leq 992 - da \leq 991\).

It remains to show that 991 is achievable. Suppose \(ab + bc + cd = 991\) and \(a = d = 1\). Then \((1 + b)(1 + c) = 992 = 2^5 \times 31\). So \(b = 30\) and \(c = 31\) is a solution. Thus the maximum value of \(ab + bc + cd\) is \(991\).

Method 2

Consider the rectangles \(a \times b\), \(b \times c\), \(c \times d\), \(a \times d\) arranged as follows. We wish to maximise the shaded area.

Let \(u = a + c\) and \(v = b + d\). For fixed \(u\) and \(v\), the shaded area is maximum when \(a = d = 1\). So, to maximise the shaded area, we need to maximise the area of the \(u \times v\) rectangle with \(u = c + 1\), \(v = b + 1\), and \(u + v = 63\).

The area of the \(u \times v\) rectangle is \(uv = u(63 - u)\). The graph of \(y = x(63 - x)\) is a parabola with its maximum at \(x = 63 / 2\). Hence the maximum value of \(uv\) is attained when \(u\) is as close as possible to 63/2. Thus \(u = 31\) and \(v = 32\) or vice versa.

So the maximum shaded area is \(31 \times 32 - 1 = 991\).

Method 3

From symmetry we may assume \(b \leq c\).

Then \(ab + bc + cd \leq ac + bc + cd = c(a + b + d) = c(63 - c)\).

The graph of \(y = x(63 - x)\) is a parabola with its maximum at \(x = 63 / 2\). Hence the maximum value of \(c(63 - c)\) is attained when \(c\) is as close as possible to 63/2. Thus \(c = 31\) or 32.

If \(b = c\), then \(a + b + c + d \geq 1 + 31 + 31 + 1 = 64\), a contradiction. So \(b < c\) and we have \(ab + bc + cd < (31)(32) = 992\).

It remains to show that 991 is achievable. If \(c = 31\), \(b = 30\), and \(a = d = 1\), then \(ab + bc + cd = 991\). Thus the maximum value of \(ab + bc + cd\) is \(991\).
8. Let $BH = u$, $HI = v$, $IC = w$, and $AF = x$.

The intersecting secant theorem at $A$ gives $4 \times 30 = x(x + 14)$. Hence $x^2 + 14x - 120 = 0$. So $(x + 20)(x - 6) = 0$ and $x = 6$. Therefore $GB = 12$. 

The intersecting secant theorem at $B$ gives $12 \times 26 = u(u + v) = u(32 - w)$. (1)

The intersecting secant theorem at $C$ gives $2 \times 28 = w(w + v) = w(32 - u)$. (2)

Subtracting (2) from (1) gives: $256 = 32(u - w)$

$u = w + 256/32 = w + 8$

$v = 32 - u - w = 24 - 2w$

Substituting in (2) gives: $56 = w(24 - w)$

$w^2 - 24w + 56 = 0$

$w = \left(24 \pm \sqrt{24^2 - 224}\right)/2$

$v = 4\sqrt{22}$

Hence $\pi b = \pi (2\sqrt{22})^2 = \pi 88$ and $b = 88$. 

9. Let the volume of the box be $V_B$ and the volume of a single tennis ball be $V_T$.

Suppose there are $N$ tennis balls to begin.

The total volume of the tennis balls is $NV_T$ and the volume of empty space surrounding them is $V_B - NV_T$.

From the ratio given, $V_B - NV_T = kNV_T$, so $V_B = NV_T + kNV_T = (k + 1)NV_T$. 

Let $P$ be the number of balls removed, where $P$ is a prime.

The total volume of the remaining tennis balls is $(N - P)V_T$ and the volume of empty space surrounding them is $V_B - (N - P)V_T = (k + 1)NV_T - (N - P)V_T = kNV_T + PV_T = (kN + P)V_T$. 

From the ratio given, $(kN + P)V_T = k^2(N - P)V_T$.

So $kN + P = k^2N - k^2P$, hence $N = P(k^2 + 1)/(k^2 - k)$. 

Now $N$ is an integer and $k$ and $k^2 + 1$ are relatively prime, so $k$ divides $P$.

But $P$ is a prime and $k > 1$ so $k = P$.

Thus $N = (P^2 + 1)/(P - 1) = P + 1 + 2/(P - 1)$. 

Since $N$ is an integer, $P - 1$ divides $2$.

So $P = 2$ or $P = 3$. Either way $N = 5$. Thus the number of tennis balls originally in the box is $5$.

Comment. The algebra can be simplified by rescaling to let $V_T = 1$. 

6
10. **Method 1**

Subdivide the square into three rectangles: one measuring $1 \text{ m} \times \frac{1}{8} \text{ m}$ and each of the other two measuring $\frac{1}{2} \text{ m} \times \frac{7}{8} \text{ m}$. 

![Diagram of the square divided into rectangles](image)

Note that $\sqrt{1 + \left(\frac{1}{8}\right)^2} = \sqrt{\frac{65}{64}}$ and $\sqrt{(\frac{1}{2})^2 + \left(\frac{7}{8}\right)^2} = \sqrt{\frac{65}{64}}$. 

So each of the three rectangles have diagonals of length $\sqrt{\frac{65}{64}}$ and can therefore be covered by a disc with this diameter.

Now $\sqrt{\frac{65}{64}} < 1.008 \iff 65 < 64(1.008)^2$ and $64(1 + 0.008)^2 > 64 \times 1.016 = 64(1 + 0.01 + 0.006) = 64 + 0.64 + 0.384 = 65.024 > 65$. 

So $\sqrt{\frac{65}{64}} < 1.008$. Therefore it is possible for three discs each with diameter 1008 mm to cover the square.
Method 2
Construct two right-angled triangles in the 1 m × 1 m square as shown.

We need to show \( d < 1008 \).

Pythagoras gives \( d^2 = 500^2 + (1000 - \sqrt{1008^2 - 1000^2})^2 \)
\[ = 500^2 + (1000 - \sqrt{(1008 - 1000)(1008 + 1000)})^2 \]
\[ = 500^2 + (1000 - \sqrt{8 \times 2008})^2 \]
\[ = 250000 + 1000000 + 16064 - 2000\sqrt{16064}. \]
So \( d^2 - 1008^2 = d^2 - 1016064 = 250000 - 2000\sqrt{16064} \).
Hence \( d < 1008 \iff 125^2 < 16064 \), which is true since \( 125^2 = 15625 \).

Investigation
The minimum diameter is \( 500\sqrt{5} \approx 1118 \text{ mm} \).

Divide the square into two rectangles each 1000 mm × 500 mm. The length of the diagonal of each rectangle is \( \sqrt{1000^2 + 500^2} = 500\sqrt{5} \text{ mm} \). Hence the square can be covered by two discs of this diameter.

Suppose the square can be covered by two discs of shorter diameter. One of the discs must cover at least two of the vertices of the square. If two of these vertices were diagonally opposite on the square, then the diameter of the disc would be at least the length of the diagonal of the square, which is approximately 1400 mm. So the disc covers exactly two vertices of the square and they are on the same side of the square.

Denote the square by \( ABCD \). We may assume the disc covers \( A \) and \( B \), and intersects \( AD \) at a point \( X \). The disc covers \( BX \) and its diameter is less than \( 500\sqrt{5} \). Hence \( BX \) is less than \( 500\sqrt{5} \), so \( AX^2 < (500\sqrt{5})^2 - 1000^2 = (125 - 100)10000 = 250000 \) and \( AX < 500 \). The second disc must cover points \( X, C, \) and \( D \). Since \( XD > 500 \), \( XC > 500\sqrt{5} \). Then the diameter of the second disc is greater than \( 500\sqrt{5} \), a contradiction.

So the diameters of the two covering discs cannot be less than \( 500\sqrt{5} \).
Marking Scheme

1. A correct approach.
   Correct answer (840).

2. Three correct combinations of digits.
   Correct answer (60).

3. Correct interpretation.
   One correct digit.
   Correct answer (248).

4. Three correct prime factorisations.
   Correct value for one prime.
   Correct maximum (42).

5. Correct units digit of the sum (1).
   Correct sum of units digits (11).
   Substantial progress.
   Correct answer (36).

6. A relevant diagram.
   Correct location of incentre on an altitude.
   Establishing a pair of similar triangles.
   Correct answer (671).

7. A correct approach.
   Further progress.
   Substantial progress.
   Correct answer (991).

8. Correct value for GB (12).
   Two other useful equations.
   A single variable expression for HI.
   Correct answer (88).

9. Correct formula for box volume in terms of ball volume.
   Correct formula for space volume in terms of ball volume.
   Correct formula in two variables for number of balls.
   Correct formula in one variable for number of balls.
   Correct answer (5).

10. A relevant diagram.
    Correct diagonal calculations.
    Further progress.
    Correct conclusion.

   Investigation:
   Establishing an upper bound for diameter of the discs.
   Establishing one disc covers two adjacent square vertices.
   Correctly establishing the conclusion.
ĐỀ THI VÀ ĐÁP ÁN CHI TIẾT

CUỘC THI VÔ ĐỊCH THÔNG KÊ TOÁN CẤP TRUNG HỌC CỦA MÔNG NGÀY 2012
1. Each letter in the grid represents a positive integer.

\[
\begin{array}{ccccccc}
  & B & C & D & E & 7 & G & H & I \\
 31 & & & & & & & \\
\end{array}
\]

The sum of any three consecutive integers in the grid is 164. Find the value of $H$.

[2 marks]

2. A yacht race takes place on the course depicted in the diagram below. The starting and finishing point is $A$, with marker buoys at $B$, $C$ and $D$.

The distance $AB$ is 6 km and the distance $DA$ is 6.5 km. Buoys $B$ and $C$ are 2 km apart and buoy $C$ is exactly the halfway mark for the race.

For a yacht moving directly from buoy $B$ to buoy $C$, its heading would be southwest. For a yacht moving directly from buoy $C$ to buoy $D$, its heading would be southeast. If the area of water bounded by the course is $R \text{km}^2$, find the value of $R$.

[2 marks]

3. Two identical bottles are filled with weak alcohol solutions. In one bottle, the ratio of the volume of alcohol to the volume of water is 1:25. In the other bottle, the ratio of the volume of alcohol to the volume of water is 1:77. If the entire contents of the two bottles are mixed together, the ratio of the volume of alcohol to the volume of water in the mixture is 1:$N$. Find the value of $N$.

[3 marks]

4. What is the maximum number of terms in a series of consecutive even positive integers whose sum is 1974?

[3 marks]
5. Let $x$ and $y$ be positive integers satisfying $x^{5x} = y^y$. What is the largest value for $x$? [4 marks]

6. The lengths of the sides of triangle $T$ are 190, 323 and 399. What is the length of the shortest altitude of $T$? [4 marks]

7. The non-zero real numbers $x, y, z$ satisfy the system of equations:

$$
xy = 2(x + y) \\
yz = 3(y + z) \\
zx = 4(z + x)
$$

Determine $5x + 7y + 9z$. [4 marks]

8. $ABCD$ is a trapezium with $AD$ parallel to $BC$. The area of $ABCD$ is 225. The area of $\triangle BPC$ is 49. What is the area of $\triangle APD$? [4 marks]

9. The $n$th triangular number is the sum of the first $n$ positive integers. Let $T_n$ denote the sum of the first $n$ triangular numbers. Derive a formula for $T_n$ and, hence or otherwise, prove that $T_n + 4T_{n-1} + T_{n-2} = n^3$. [5 marks]

10. A bag contains a certain number of 5-cent, 10-cent, 20-cent, 50-cent and one-dollar coins of more than one denomination. For example, if the bag contains nine 5-cent coins and one 20-cent coin, then it contains exactly ten coins and two denominations. Notice in this example that, if any single coin is removed from the bag, then the remaining coins may be divided into three heaps of equal value. Determine all possible combinations of distinct denominations the bag may contain so that after removing any coin from the bag, its contents can be divided into three heaps of equal value. [4 marks]

*Investigation*

Suppose the bag contains a certain number of 5-cent, 10-cent, 20-cent, 50-cent, one-dollar, and two-dollar coins of more than one denomination. Determine all possible combinations of distinct denominations the bag may contain so that after removing any coin from the bag, its contents can be divided into two heaps of equal value. [4 bonus marks]
Australian Intermediate Mathematics Olympiad 2012

Solutions

1. Method 1

\[ 31 + B + C = 164 \text{ and } B + C + D = 164, \text{ so } D = 31. \]
\[ E + 7 + G = 164 \text{ and } 7 + G + H = 164, \text{ so } E = H. \]
\[ D + E + 7 = 164 \text{ and } D = 31, \text{ so } H = E = 164 - 7 - 31 = 126. \]

Method 2

The sum of any three consecutive entries in the grid is a constant, so we have the following pattern

| x | y | z | x | y | z | x | y | z |

Given that \( x = 31, z = 7, \) and \( x + y + z = 164, \) we have \( H = y = 164 - 31 - 7 = 126. \)

2. From the information about distances (\( C \) is halfway), \( CD = 8 - 6.5 = 1.5 \text{ km}. \)

From the information about headings, \( \angle BCD \) is a right angle.

In \( \triangle BCD, BD = 2.5 \text{ km} \) (3-4-5 triangle, or use Pythagoras' Theorem directly).

Method 1

In \( \triangle ABD, \angle ABD \) is a right angle (converse of Pythagoras' Theorem in 5-12-13 triangle).

Area of water = area \( \triangle ABD \) - area \( \triangle BCD = \frac{1}{2}(6 \times 2.5 - 2 \times 1.5) = 6 \text{ km}^2. \) So \( R = 6. \)

Method 2

Half the perimeter of \( \triangle ABD \) is 7.5 km. From Heron’s formula, the square of the area of \( \triangle ABD \) is \( 7.5(7.5 - 6)(7.5 - 6.5)(7.5 - 2.5) = 7.5(1.5)(1.5) = 7.5^2. \)

Half the perimeter of \( \triangle BCD \) is 3 km. From Heron’s formula, the square of the area of \( \triangle BCD \) is \( 3(3 - 2.5)(3 - 2)(3 - 1.5) = 3(0.5)(1)(1.5) = 1.5^2. \)

Area of water = area \( \triangle ABD \) - area \( \triangle BCD = 7.5 - 1.5 = 6 \text{ km}^2. \) So \( R = 6. \)

3. Let \( V \) be the volume of one bottle.

The volume of alcohol in the first bottle is \( \frac{1}{25}V. \)

The volume of alcohol in the second bottle is \( \frac{1}{75}V. \)

The proportion of alcohol by volume in the combined mixture is

\[ \frac{\frac{1}{25}V + \frac{1}{75}V}{2V} = \frac{1}{75}(1 + \frac{1}{3}) = \frac{1}{75} \times \frac{4}{3} = \frac{1}{39}. \]

Thus the ratio of volume of alcohol to volume of water is 1:38. Hence \( N = 38. \)
4. **Method 1**

Let \( n \) be the number of terms in the series.

If the series starts with 2, then \( 1974 = \frac{n}{2}(2 + 2n) = n(n + 1) \).
This has no integer solution since \( 43 \times 44 = 1892 < 1974 \) and \( 44 \times 45 = 1980 > 1974 \).

If the series starts with 4, then \( 1974 = \frac{n}{2}(4 + 2n + 2) = n(n + 3) \).
This has no integer solution since \( 42 \times 45 = 1890 < 1974 \) and \( 43 \times 46 = 1978 > 1974 \).

If the series starts with 6, then \( 1974 = \frac{n}{2}(6 + 2n + 4) = n(n + 5) \).
This has integer solution \( n = 42 \).

If the first term is increased, then the number of terms must decrease to get the same sum.
So the maximum number of terms is **42**.

**Method 2**

Let \( n \) be the number of terms in the series.

We have \( 1974 = 2a + (2a + 2) + \ldots + (2a + 2n - 2) = \frac{n}{2}(4a + 2n - 2) = n(2a + n - 1) \).

Now \( 1974 = 2 \times 3 \times 7 \times 47 \).
So \( n \) is one of the factors 1, 2, 3, 6, 7, 14, 21, 42, 47, \ldots

As \( n \) increases, \( 2a + n - 1 \) decreases and hence \( a \) decreases.
If \( n = 42 \) then \( 2a + n - 1 = 1974 \div 42 = 47 \), hence \( 2a = 6 \).
If \( n = 47 \) then \( 2a + n - 1 = 1974 \div 47 = 42 \), hence \( 2a \) is negative.
So the maximum number of terms is **42**.

**Method 3**

Let \( n \) be the number of terms in the series and let \( A \) be their average.

Then \( nA = 1974 = 2 \times 3 \times 7 \times 47 \).
So \( n \) is one of the factors 1, 2, 3, 6, 7, 14, 21, 42, 47, \ldots

Now, if \( n \) is odd then \( A \) is the middle term in the series and if \( n \) is even then \( A \) is the odd integer between the two middle terms of the series.

If \( n = 42 \), then \( A = 47 \) and the first term is \( 47 + 1 - 21(2) = 6 \).
If \( n = 47 \), then \( A = 42 \) and the first term is \( 42 - 23(2) < 0 \).
As \( n \) increases, \( A \) decreases, hence the first term remains negative.
So the maximum number of terms is **42**.
5. Method 1

Clearly $x = 1$ satisfies the given equation.

Suppose $x > 1$ and $x^{5x} = y^9$. (*)

If $y \leq x$, then $y^9 \leq x^x < x^{5x}$, which is a contradiction. Hence $x < y$.

Now suppose $5x \leq y$. Then $y^9 \geq (5x)^{5x} = 5^{5x}x^{5x}$, again a contradiction. So $x < y < 5x$. 1

Let $p$ be any prime factor of $x$. By (*), $p$ divides $y$.

So $x = p^a x'$ and $y = p^b y'$, where $a$ and $b$ are positive integers and $p$ divides neither $x'$ nor $y'$.

By (*), $5ax = yb$, so that $a = \frac{x}{5}b < b$. Hence $p^a$ divides $p^b$.

Since $p$ is an arbitrary prime factor of $x$, $x$ divides $y$.

It follows that $y = cx$, with $c = 2, 3, \text{or } 4$.

Feeding this back into (*), we see that $x^{5x} = (cx)^{5x}$, whence $x^5 = (cx)^c$, or $x^{5-c} = c^c$.

If $c = 2$, then $x^3 = 4$, which is impossible.

If $c = 3$, then $x^2 = 27$, again impossible.

If $c = 4$, then $x = 4^4 = 256$.

Now $256^{5 \times 256} = 2^{8 \times 5 \times 256} = 2^{10 \times 1024} = 1024^{1024}$.

Thus $x = 256$ is the only value for $x$, other than 1, that satisfies $x^{5x} = y^9$.

Hence the largest value for $x$ is $256$. 1

Method 2

Clearly $x = 1$ satisfies the given equation. Suppose $x > 1$.

Then $x < y$, otherwise $y^9 \leq x^x < x^{5x}$, which is a contradiction.

Since $x^9 = x^{5x}$, $x$ and $y$ have the same prime factors.

Let $x = p_1^{s_1}p_2^{s_2}\cdots$ and $y = p_1^{t_1}p_2^{t_2}\cdots$.

Then $5s_i x = t_i y$ for all $i$. So $t_i = k s_i$ for all $i$ where $k$ is a constant.

Since $x < y$, $k > 1$.

Suppose $x$ has at least two distinct primes.

Then $5 > y/x > p_1^{1-s_1}p_2^{1-s_2} \geq (2)(3) = 6$, which is a contradiction.

Therefore $x = p^2$ and $y = p^4$, where $p$ is prime and $s < t$.

We have $5sp^4 = tp^4$, and $5s = tp^{4-s} > sp^{4-s}$, hence $5 > p^{4-s}$.

Therefore $p = 3$ and $t - s = 1$ or $p = 2$ and $t - s = 1$ or 2.

If $p = 3$ and $t - s = 1$, then $5s = 3t = 3s + 3$ and $s$ is not an integer.

If $p = 2$ and $t - s = 1$, then $5s = 2t = 2s + 2$ and $s$ is not an integer.

If $p = 2$ and $t - s = 2$, then $5s = 4t = 4s + 8, s = 8, x = 2^8$, and $y = 2^{10}$.

Since $(2^8)^{5 \times 2^8} = 2^{5 \times 2^{11}} = (2^{10})^{2^{10}}$, the largest value for $x$ is $2^8 = 256$. 1
6. **Method 1**

Since $190 = 19 \times 10$, $323 = 19 \times 17$, and $399 = 19 \times 21$, we first rescale triangle $T$ by dividing all lengths by 19. This simplifies the calculations. The shortest altitude in a triangle is perpendicular to its longest side. Let $t$ be the length of the shortest altitude and divide the longest side into lengths $p$ and $q$ as shown.

Alternative 1

Pythagoras gives

$17^2 = p^2 + t^2$ and $10^2 = q^2 + t^2 = (21 - p)^2 + t^2 = 21^2 - 42p + p^2 + t^2$.

Hence $42p = 17^2 + 21^2 - 10^2 = 289 + 441 - 100 = 630$.

So $p = 15$ and $t^2 = 17^2 - 15^2 = 64$.

Thus $t = 8$. Hence the shortest altitude in $T$ is $8 \times 19 = 152$.

Alternative 2

Let $x$ be the angle in the bottom-right corner of $T$.

From the cosine rule, $17^2 = 10^2 + 21^2 - 2(10)(21) \cos x$.

Hence $\cos x = (100 + 441 - 289)/420 = 252/420 = 21/35 = 3/5$. So $q = 10 \cos x = 6$.

Pythagoras gives $t^2 = 100 - 36 = 64$.

Thus $t = 8$. Hence the shortest altitude in $T$ is $8 \times 19 = 152$.

**Method 2**

Hal the perimeter of $T$ is $(190 + 323 + 399)/2 = 912/2 = 456$.

From Heron’s formula, the square of the area of triangle $T$ is


The shortest altitude in a triangle is perpendicular to its longest side.

If $t$ is the length of the shortest altitude of $T$, then the area of $T$ is $\frac{1}{2}(399)t = 4(57)(133)$.

So $3t = 8(57)$ and $t = 8(19) = 152$. 

7. Method 1

We rewrite the system as:

\[ 1 = 2 \left( \frac{1}{x} + \frac{1}{y} \right), \quad \frac{1}{x} = 3 \left( \frac{1}{y} + \frac{1}{z} \right), \quad 1 = 4 \left( \frac{1}{z} + \frac{1}{x} \right). \]  \hspace{1cm} (1)

Put \( X = \frac{1}{x}, \ Y = \frac{1}{y}, \ Z = \frac{1}{z} \). Then

\[ X + Y = \frac{1}{x}, \quad Y + Z = \frac{1}{y}, \quad Z + X = \frac{1}{z}. \]  \hspace{1cm} (1)

Adding these equations yields

\[ X + Y + Z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{13}{24}. \]  \hspace{1cm} (1)

Hence

\[ X = \frac{13}{24} - \frac{3}{4} = \frac{5}{24}, \quad Y = \frac{13}{24} - \frac{1}{4} = \frac{7}{24}, \quad Z = \frac{13}{24} - \frac{1}{2} = \frac{5}{24}. \]  \hspace{1cm} (1)

Therefore \( 5x + 7y + 9z = 24 + 24 + 216 = 264 \).

Method 2

We rewrite the system as:

\[ \frac{1}{x} = \frac{1}{2} + \frac{1}{y}, \quad \frac{1}{y} = \frac{1}{2} + \frac{1}{z}, \quad \frac{1}{z} = \frac{1}{2} + \frac{1}{x}. \]  \hspace{1cm} (1)

Adding and subtracting these equations gives:

\[ \frac{1}{2} - \frac{1}{y} + \frac{1}{x} = \frac{3}{4}, \quad x = 2 \left( \frac{12}{y - 4} \right) = \frac{24}{y}. \]  \hspace{1cm} (1)

\[ \frac{1}{2} - \frac{1}{y} + \frac{1}{z} = \frac{2}{3}, \quad y = 2 \left( \frac{12}{y - 4} \right) = \frac{24}{y} \]  \hspace{1cm} (1)

\[ \frac{1}{2} - \frac{1}{y} + \frac{1}{x} = \frac{5}{3}, \quad z = 2 \left( \frac{12}{y - 4} \right) = 24 \]  \hspace{1cm} (1)

Therefore \( 5x + 7y + 9z = 24 + 24 + 216 = 264 \).

Method 3

We have

\[ xy = 2(x + y) \quad (1) \]
\[ yz = 3(y + z) \quad (2) \]
\[ zx = 4(z + x) \quad (3) \]

Since no variable is zero, equation (1) implies \( y \neq 2 \), equation (2) implies \( y \neq 3 \), and equation (3) implies \( z \neq 4 \).

From (1), \( x = \frac{2y}{y - 2} \). From (2), \( z = \frac{3y}{y - 3} \). From (3), \( x = \frac{4z}{z - 4} \).  \hspace{1cm} (1)

Hence \( \frac{2y}{y - 2} = \frac{4z}{z - 4} = \frac{12y}{y - 3} = \frac{12y}{y - 3} \)  \hspace{1cm} (1)

So \( 24y - 2y^2 = 12y^2 - 24y \), \( 48y = 14y^2 \), \( y = \frac{24}{y} \).

Then \( x = \frac{48}{24 - 14} = \frac{24}{7} \) and \( z = \frac{72}{24 - 21} = 24 \).  \hspace{1cm} (1)

Therefore \( 5x + 7y + 9z = 24 + 24 + 216 = 264 \).  \hspace{1cm} (1)
**Method 4**

We have

\[ xy = 2(x + y) \]  
\[ yz = 3(y + z) \]  
\[ zx = 4(z + x) \]

Hence

\[ xyz = 2z(x + y) = 2xz + 2yz = 8z + 8x + 6y + 6z = 8x + 6y + 14z \]  
\[ xyz = 3xy + 3xz = 3x + 3y + 6z + 6y + 12z + 12x = 18x + 6y + 12z \]  
\[ xyz = 4yz + 4xy = 12y + 12y + 8x + 8y = 8x + 20y + 12z \]

From (4) and (5), \( z = 5x \). From (4) and (6), \( z = 7y \).

From (3), \( 5x^2 = 24x, x = 24/5 \). From (2), \( 7y^2 = 24y, y = 24/7 \), \( z = 24 \).

Therefore \( 5x + 7y + 9z = 24 + 24 + 216 = 264 \).

8. Let \( || \) denote area.

Since \( AD \) is parallel to \( BC \), triangles \( APD \) and \( CPB \) are similar.

**Method 1**

We have

\[ \frac{PD}{PB} = \frac{PA}{PC} = \sqrt{\frac{\triangle APD}{\triangle CPB}}. \]

\[ |\triangle PDC| = \frac{PD}{PB} \times |\triangle PBC| = \frac{\sqrt{|\triangle APD|}}{7} \times 49 = 7 \times \sqrt{|\triangle APD|}. \]

\[ |\triangle PAB| = \frac{PA}{PC} \times |\triangle PCB| = \frac{\sqrt{|\triangle APD|}}{7} \times 49 = 7 \times \sqrt{|\triangle APD|}. \]

Therefore \( 225 = 49 + 14\sqrt{|\triangle APD|} + |\triangle APD| = (7 + \sqrt{|\triangle APD|})^2 \).

So \( 15 = 7 + \sqrt{|\triangle APD|} \). Hence \( |\triangle APD| = (15 - 7)^2 = 64 \).

**Method 2**

We have

\[ \frac{PD}{PB} = \frac{PA}{PC} = \frac{AD}{CB} = k, \text{ say.} \]

\[ |\triangle CPD| = \frac{1}{2} CP \times PD \sin \angle CPD = \frac{1}{2} BP \times PA \sin \angle BPA = |\triangle BPA| = R, \text{ say.} \]

Let \( |\triangle APD| = T \).

We have:

\[ 225 = 49 + T + 2R, \]
\[ T/R = AP/PC = k, \]
\[ T/49 = \frac{1}{2} AP \times PD \sin \angle APC)/(\frac{1}{2} BP \times PC \sin \angle BPC) = k^2. \]

Hence \( 225 = 49 + 49k^2 + 2(49k) = 49(1 + 2k + k^2) = 49(1 + k)^2 \).

So \( k = 15/7 - 1 = 8/7 \). Hence \( T = 49(8/7)^2 = 64 \).
9. Part 1. First we find a formula for \( T_n \).

Method 1

\[
T_n = 1 + 3 + 6 + \cdots + \frac{n}{2}(n + 1)
\]

\[
2T_n = 2 + 6 + 12 + \cdots + n(n + 1)
\]

\[
= (1)(1 + 1) + (2)(2 + 1) + (3)(3 + 1) + \cdots + (n)(n + 1)
\]

\[
= (1 + 4 + 9 + \cdots + n^2) + (1 + 2 + 3 + \cdots + n)
\]

\[
= \frac{n}{6}(2n + 1)(n + 1) + \frac{n}{2}(n + 1)
\]

\[
= \frac{n(n + 1)(2n + 1 + 3)}{6}
\]

So \( T_n = \frac{n(n + 1)(n + 2)}{6} \).

Method 2

\[
T_n = 1 + 3 + 6 + \cdots + \frac{n}{2}(n + 1)
\]

\[
6T_n = 6 + 18 + 36 + \cdots + 3n(n + 1)
\]

\[
= (2^3 - 1^3 - 1) + (3^3 - 2^3 - 1) + (4^3 - 3^3 - 1) + \cdots + ((n + 1)^3 - n^3 - 1)
\]

\[
= (2^3 - 1^3) + (3^3 - 2^3) + (4^3 - 3^3) + \cdots + ((n + 1)^3 - n^3) - n
\]

\[
= (n + 1)^3 - n - 1
\]

\[
= (n + 1)((n + 1)^2 - 1) = (n + 1)n(n + 2)
\]

So \( T_n = \frac{n(n + 1)(n + 2)}{6} \).

Method 3

We represent triangular numbers by rectangles of width 1. Packing these rectangles with squares gives another rectangle. The area of this rectangle gives a formula for \( T_n \).

From the area of the large rectangle, we can see that

\[
(1^2 + 2^2 + 3^2 + \cdots + n^2) + T_n = (1 + 2 + \cdots + n)(n + 1) = \frac{n}{2}(n + 1)(n + 1).
\]

Hence \( T_n = \frac{1}{2}n(n + 1)(n + 1) - \frac{1}{2}(n + 1)(2n + 1) \)

\[
= \frac{n}{6}(n + 1)(3n + 3 - 2n - 1)
\]

\[
= \frac{n}{6}(n + 1)(n + 2).
\]
Method 4

The \( n \)th triangular number is \( \frac{n}{2}(n+1) \), a quadratic polynomial in \( n \). This suggests that \( T_n \) is a cubic polynomial in \( n \), that is, \( T_n = an^3 + bn^2 + cn + d \) where \( a, b, c, d \) are constants that we have to calculate.

We have:

\[
\begin{align*}
T_1 &= a + b + c + d &= 1 \\
T_2 &= 8a + 4b + 2c + d &= 4 \\
T_3 &= 27a + 9b + 3c + d &= 10 \\
T_4 &= 64a + 16b + 4c + d &= 20
\end{align*}
\]

Subtracting equation (3) from (4), then (2) from (3), then (1) from (2) gives:

\[
\begin{align*}
a + b + c + d &= 1 \\
7a + 3b + c + 0 &= 3 \\
26a + 8b + 2c + 0 &= 9 \\
63a + 15b + 3c + 0 &= 19
\end{align*}
\]

Subtracting equation (6) from (7) twice, then (6) from (8) three times gives:

\[
\begin{align*}
a + b + c + d &= 1 \\
7a + 3b + c + 0 &= 3 \\
12a + 2b + 0 + 0 &= 3 \\
42a + 6b + 0 + 0 &= 10
\end{align*}
\]

Subtracting equation (11) from (12) three times gives:

\[
\begin{align*}
a + b + c + d &= 1 \\
7a + 3b + c + 0 &= 3 \\
12a + 2b + 0 + 0 &= 3 \\
6a + 0 + 0 + 0 &= 1
\end{align*}
\]

From (16) \( a = \frac{1}{6} \). From (15) \( b = \frac{3-2}{2} = \frac{1}{2} \).

From (14) \( c = 3 - \frac{7}{6} - \frac{3}{2} = \frac{1}{4} \). From (13) \( d = 1 - \frac{1}{4} - \frac{1}{2} - \frac{1}{6} = 0 \).

So \( T_n = \frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3} = \frac{n(n^2 + 3n + 2)}{6} = \frac{n(n+1)(n+2)}{6} \).

We know from our calculations that this formula gives the first four tetrahedral numbers. To prove it gives all tetrahedral numbers, we show that the difference \( T_n - T_{n-1} \) is the \( n \)th triangular number:

\[
T_n - T_{n-1} = \frac{n(n+1)(n+2)}{6} - \frac{(n-1)n(n+1)}{6} = \frac{n(n+1)(n+2 - n)}{6} = \frac{n(n+1)}{2}.
\]
Method 5
The \( n \)th triangular number is \( \frac{n}{2}(n + 1) \), a quadratic polynomial in \( n \). This suggests that \( T_n \) is a cubic polynomial in \( n \), that is, \( T_n = an^3 + bn^2 + cn + d \) where \( a, b, c, d \) are constants that we have to calculate.

We have:
\[
T_n - T_{n-1} = \frac{n}{2}(n + 1).
\]

Therefore:
\[
\begin{align*}
\frac{1}{2}n^2 + \frac{1}{2}n &= an^3 + bn^2 + cn + d - (a(n-1)^3 + b(n-1)^2 + c(n-1) + d) \\
&= an^3 + bn^2 + cn - (a(n^3 - 3n^2 + 3n - 1) + b(n^2 - 2n + 1) + c(n-1)) \\
&= 3an^2 - 3an + a + 2bn - b + c
\end{align*}
\]

Equating corresponding coefficients of powers of \( n \) we get:
\[
3a = \frac{1}{2}, \quad -3a + 2b = \frac{1}{2}, \quad a - b + c = 0.
\]

So \( a = \frac{1}{6}, \quad b = \frac{1}{2}, \quad c = \frac{1}{2} - \frac{1}{6} = \frac{1}{3} \).

Hence \( T_n = \frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3} + d \).

Therefore \( T_1 = \frac{1}{6} + \frac{1}{2} + \frac{1}{3} + d = 1 + d \). Since \( T_1 = 1, \quad d = 0 \).

So \( T_n = \frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3} = \frac{n(n+1)(n+2)}{6} \).

Method 6
From the construction of Pascal’s Triangle, the second column gives \( n \), the third column gives the \( n \)th triangular number, and the fourth column gives the sum of the first \( n \) triangular numbers.

\[
\begin{array}{cccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\
1 & 8 & 28 & 46 & 70 & 46 & 28 & 8 & 1 \\
1 & 9 & 36 & 74 & 116 & 116 & 74 & 36 & 9 & 1 \\
\end{array}
\]

Thus \( T_n = \binom{n+2}{3} = \frac{n(n+1)(n+2)}{6} \).
Part 2. Next we prove that $T_n + 4T_{n-1} + T_{n-2} = n^3$.

**Method 1**

\[
T_n + 4T_{n-1} + T_{n-2} = \frac{n(n+1)(n+2)}{6} + 4 \frac{(n-1)(n)(n+1)}{6} + \frac{(n-2)(n-1)(n)}{6} = \frac{n}{6}((n+1)(n+2) + 4(n-1)(n+1) + (n-2)(n-1)) = \frac{n}{6}(6n^2 + 3n + 2 - 4 - 3n + 2) = n^3
\]

**Method 2**

This is a geometric proof. We represent the $5 \times 5 \times 5$ cube by five $5 \times 5$ horizontal layers, top layer first and bottom layer last. The numbers represent the six required tetrahedra: tetrahedon 1 is $T_3$, tetrahedra 2 to 5 are $T_4$, and tetrahedron 6 is $T_5$. Thus $T_5 + 4T_4 + T_3 = 5^3$.

\[
\begin{array}{cccccccccc}
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
5 & 5 & 5 & 5 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 5 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
5 & 5 & 5 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 5 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
5 & 5 & 5 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 5 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
5 & 5 & 5 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 5 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
5 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
\end{array}
\]

The diagrams clearly generalise for all $n \geq 3$. 

1
Finally, we can see that 

\[ 2(1 + 2 + 3 + 4) = 4(4 + 1). \]

For general \( n \), these can be established in various ways. For example, here is a geometrical proof for each.

First consider summing the first \( n \) integers. Each number \( k \) is represented by a \( k \times 1 \) rectangle. Take \( n = 4 \) as an example. From the area of the large rectangle in the following diagram, we can see that 

\[ 2(1 + 2 + 3 + 4) = 4(4 + 1). \]

For general \( n \) we get 

\[ 1 + 2 + \cdots + n = \frac{1}{2}n(n + 1). \]

| \[ \begin{array}{ll} 
   4 & 1 \\
   3 & 2 \\
   2 & 3 \\
   1 & 4 \\
\end{array} \] |

In a similar way we can find a formula for the sum of the first \( n \) squares. Again, take \( n = 4 \) as an example.

| \[ \begin{array}{llll} 
   1 & 2 & 3 & 4 \\
   1 & 2 & 3 & 4 \\
   1 & 2 & 9 & 16 \\
   1 & 4 & 9 & 16 \\
\end{array} \] |

From the area of the large rectangle, we can see that 

\[ (1 + 2 + 3 + 4)(4 + 1) = (1^2 + 2^2 + 3^2 + 4^2) + (1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4)). \]

In general, 

\[ (1 + 2 + \cdots + n)(n + 1) = (1^2 + 2^2 + \cdots + n^2) + (1 + (1 + 2) + \cdots + (1 + 2 + \cdots + n)). \]

Thus \( \frac{1}{6}n(n + 1) = (1^2 + 2^2 + \cdots + n^2) + 1 + \frac{1}{2}(2 + 1) + \cdots + \frac{1}{6}n(n + 1). \)

Hence \( n(n + 1)^2 = 2(1^2 + 2^2 + \cdots + n^2) + (1 + 1) + (2^2 + 2) + \cdots + (n^2 + n) \)

\[ = 3(1^2 + 2^2 + \cdots + n^2) + (1 + 2 + \cdots + n) \]

\[ = 3(1^2 + 2^2 + \cdots + n^2) + \frac{1}{2}n(n + 1). \]

So \( 6(1^2 + 2^2 + \cdots + n^2) = 2n(n + 1)^2 - n(n + 1) \)

\[ = n(n + 1)(2(n + 1) - 1) \]

\[ = n(n + 1)(2n + 1). \]

Finally \( 1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n + 1)(2n + 1). \)
10. Let the total value of the contents of the bag be $S$ cents. Remove one coin of denomination $m$ cents, say. Then the total value of the coins left in the bag is $S - m$ cents. This amount must be divisible by 3. Similarly, if one coin of denomination $n$ cents is removed, $S - n$ must be divisible by 3. Hence $(S - n) - (S - m) = m - n$ must also be divisible by 3. So any two denominations occurring in the bag differ by a multiple of 3, hence each must have the same remainder when divided by 3.

The available denominations are: 5, 10, 20, 50, 100 cents. They fall into two classes according to their remainders after division by 3: \{5, 20, 50\} and \{10, 100\}.

The following table shows, with examples, all combinations of denominations that could have occurred in the bag.

<table>
<thead>
<tr>
<th>Denominations</th>
<th>Possible coins</th>
<th>One coin removed</th>
<th>Three equal heaps</th>
</tr>
</thead>
<tbody>
<tr>
<td>5, 20</td>
<td>1 × 20, 9 × 5</td>
<td>9 × 5</td>
<td>3 × (3 × 5)</td>
</tr>
<tr>
<td>5, 50</td>
<td>1 × 50, 21 × 5</td>
<td>21 × 5</td>
<td>3 × (7 × 5)</td>
</tr>
<tr>
<td>20, 50</td>
<td>4 × 50, 6 × 20</td>
<td>3 × 50, 6 × 20</td>
<td>3 × (1 × 50 + 2 × 20)</td>
</tr>
<tr>
<td>5, 20, 50</td>
<td>2 × 50, 30 × 5</td>
<td>1 × 50, 2 × 20, 30 × 5</td>
<td>1 × 50 + 2 × 5, 2 × 5, 1 × 20 + 12 × 5, 16 × 5</td>
</tr>
<tr>
<td>10, 100</td>
<td>1 × 100, 21 × 10</td>
<td>21 × 10</td>
<td>3 × (7 × 10)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1 × 100, 20 × 10</td>
<td>1 × 100, 10 × 10, 10 × 10</td>
</tr>
</tbody>
</table>
Investigation

Let the total value of the contents of the bag be $S$ units. Remove one coin of denomination $m$ units, say. Then the total value of the coins left in the bag is $S - m$ units. This amount must be divisible by 2. Similarly, if one coin of denomination $n$ units is removed, $S - n$ must be divisible by 2. Hence $(S - n) - (S - m) = m - n$ must also be divisible by 2. So all the denominations in the bag must be even or all must be odd.

Since there are at least two denominations in the bag and the only odd denomination is 5 cents, there cannot be any 5 cent coins in the bag. This leaves us with the following set of denominations: 10, 20, 50, 100 and 200 cents. All of them are multiples of 10, so 10 cents is our new unit. Call it a bob, both singular and plural. The denominations are now 1, 2, 5, 10 and 20 bob. Hence the denominations in the bag will be 1 bob and 5 bob (both odd), or at least two of 2, 10, and 20 bob (all even).

In the latter case all three denominations are multiples of two, so our new unit is two-bob (originally 20 cents). Thus each coin in this bag will be worth 1, 5, or 10 two-bobs. Since 10 is the only even number amongst 1, 5, and 10, it cannot be a denomination occurring in the bag.

Hence the denominations in the bag will be 1 bob and 5 bob (i.e. 10 cents and 50 cents) or 1 two-bob and 5 two-bobs (i.e. 20 cents and 100 cents). Here are some examples to show both are possible.

<table>
<thead>
<tr>
<th>Denominations</th>
<th>Possible coins</th>
<th>One coin removed</th>
<th>Two equal heaps</th>
</tr>
</thead>
<tbody>
<tr>
<td>10, 50</td>
<td>1 × 50, 6 × 10</td>
<td>6 × 10</td>
<td>2 × (3 × 10)</td>
</tr>
<tr>
<td></td>
<td>1 × 50, 5 × 10</td>
<td>1 × 50, 5 × 10</td>
<td></td>
</tr>
<tr>
<td>20, 100</td>
<td>1 × 100, 6 × 20</td>
<td>6 × 20</td>
<td>2 × (3 × 20)</td>
</tr>
<tr>
<td></td>
<td>1 × 100, 5 × 20</td>
<td>1 × 100, 5 × 20</td>
<td></td>
</tr>
</tbody>
</table>
Marking Scheme

1. A correct approach.
   Correct answer (126).

2. Correct distance for BD (2.5 km).
   Correct answer (6).

3. Correct proportions of alcohol in two vessels (V/26, V/78).
   Correct proportion of alcohol in combined mixture (1/39).
   Correct value for N (38).

4. A correct approach.
   A correct list of options or cases for n.
   Correct maximum (42).

5. A correct approach.
   Substantial progress.
   Completion of cases.
   Correct maximum value (256).

6. A relevant diagram or correct semiperimeter.
   A useful formula correctly applied.
   Another useful formula correctly applied.
   Correct answer (152).

7. A relevant system of equations.
   A useful set of substitutions.
   Correct values for x, y, z (24/5, 24/7, 24).
   Correct answer (264).

8. A correct approach.
   Establishing areas of triangles CPD and BPA equal.
   Relevant equation in one variable.
   Correct area of triangle APD (64).

   Substantial progress.
   Correct formula for $T_n = \frac{n(n + 1)(n + 2)}{6}$.
   A correct approach to proving $T_n + 4T_{n-1} + T_{n-2} = n^3$.
   A convincing proof.

10. Establishing that any two denominations are congruent modulo 3.
    Two correct congruency classes.
    Demonstrating three denomination combinations are possible.
    Demonstrating the remaining two denomination combinations are possible.

   **Investigation:**
   Establishing that any two denominations are congruent modulo 2.
   Two correct congruency classes after dividing by 10.
   Two correct congruency classes after further dividing by 2.
   Demonstrating the two denomination combinations are possible.  
   
   | bonus 1 | bonus 1 | bonus 1 |